Verification and Strengthening of the Atiyah–Sutcliffe Conjectures for Several Types of Configurations

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Abstract

In 2001 Sir M. F. Atiyah formulated a conjecture C1 and later with P. Sutcliffe two stronger conjectures C2 and C3. These conjectures, inspired by physics (spin-statistics theorem of quantum mechanics), are geometrically defined for any configuration of points in the Euclidean three space. The conjecture C1 is proved for $n = 3, 4$ and for general n only for some special configurations (M. F. Atiyah, M. Eastwood and P. Norbury, D.Đoković). Interestingly the conjecture C2 (and also stronger C3) is not yet proven even for arbitrary four points in a plane. So far we have verified the conjectures C2 and C3 for parallelograms, cyclic quadrilaterals and some infinite families of tetrahedra.

We have also proposed a strengthening of conjecture C3 for configurations of four points (Four Points Conjectures).

For almost collinear configurations (with all but one point on a line) we propose several new conjectures (some for symmetric functions) which imply C2 and C3. By using computations with multi-Schur functions we can do verifications up to $n = 9$ of our conjectures. We can also verify stronger conjecture of Đoković which imply C2 for his nonplanar configurations with dihedral symmetry.

Finally we mention that by minimizing a geometrically defined energy, figuring in these conjectures, one gets a connection to some complicated physical theories, such as Skyrmions and Fullerenes.

1 Introduction on Geometric Energies

In this Section we describe some geometric energies, introduced by Atiyah. To construct first geometric energy consider n distinct ordered points, $\mathbf{x}_i \in \mathbb{R}^3$ for $i = 1, ..., n$. For each pair $i \neq j$ define the unit vector

$$\mathbf{v}_{ij} = \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|} \tag{1.1}$$

giving the direction of the line joining $\mathbf{x}_i$ to $\mathbf{x}_j$. Now let $t_{ij} \in \mathbb{CP}^1$ be the point on the Riemann sphere associated with the unit vector $\mathbf{v}_{ij}$, via the identification $\mathbb{CP}^1 \cong S^2$, realized as stereographic projection. Next, set p_i to be the polynomial in t with roots t_{ij} $(j \neq i)$, that is

$$p_i = \alpha_i \prod_{j \neq i} (t - t_{ij}) \tag{1.2}$$

where α_i is a certain normalization coefficient. In this way we have constructed n polynomials which all have degree $n - 1$, and so we may write

$$p_i = \sum_{j=1}^{n} m_{ij} t^{j-1}.$$

Finally, let M_n be the $n \times n$ matrix with entries m_{ij}, and let $\mathcal{D}_n$ be its determinant

$$\mathcal{D}_n = \mathcal{D}_n(\mathbf{x}_1, ..., \mathbf{x}_n) = \det M_n. \tag{1.3}$$

This geometrical construction is relevant to the Berry-Robbins problem, which is concerned with specifying how a spin basis varies as n point particles move in space, and supplies a solution provided it can be shown that $\mathcal{D}_n$ is always non-zero. For $n = 2, 3, 4$ it can be proved that $\mathcal{D}_n \neq 0$ (Atiyah $n = 3$, Eastwood and Norbury $n = 4$) and numerical computations suggest that $|\mathcal{D}_n| \geq 1$ for all n, with the minimal value $|\mathcal{D}_n| = 1$ being attained by n collinear points.

The geometric energy is the n-point energy defined by

$$E_n = -\log |\mathcal{D}_n|, \tag{1.4}$$

so minimal energy configurations maximize the modulus of the determinant.

This energy is geometrical in the sense that it only depends on the directions of the lines joining the points, so it is translation, rotation and scale invariant. Remarkably, the minimal energy configurations, studied numerically for all $n \leq 32$, are essentially the same as those for the Thomson problem.

2 Eastwood–Norbury formulas for Atiyah determinants

In this section we first recall Eastwood–Norbury formula for Atiyah determinant for three or four points in Euclidean three–space. In the case $n = 3$ the (non

normalized) Atiyah determinant reads as

$$D_3 = d_3(r_{12}, r_{13}, r_{23}) + 8r_{12}r_{13}r_{23}$$

where

$$d_3(a, b, c) = (a + b - c)(b + c - a)(c + a - b)$$

and r_{ij} $(1 \le i < j \le 3)$ is the distance between the i^{th} and j^{th} point.

The normalized Atiyah determinant for 3 points is

$$\mathcal{D}_3 = \frac{D_3}{8r_{12}r_{13}r_{23}}$$

and it is evident that $|\mathcal{D}_3| = \mathcal{D}_3 \ge 1$.

In the case $n = 4$ the (non normalized) Atiyah determinant D_4 has real part given by a polynomial (with 248 terms) as follows:

$$\Re(D_4) = 64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} - 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) + A_4 + 288V^2 \quad (2.5)$$

where

$$A_4 = \sum_{l=1}^{4} \left(\sum_{(l \neq) i=1}^{4} r_{li}((r_{lj} + r_{lk})^2 - r_{jk}^2) \right) d_3(r_{ij}, r_{ik}, r_{jk})$$

(here $\{j, k\} = \{1, 2, 3, 4\} \setminus \{l, i\}$) and V denotes the volume of the tetrahedron with vertices our four points:

$$\begin{aligned} 144V^2 = \quad & r_{12}^2 r_{34}^2 (r_{13}^2 + r_{14}^2 + r_{23}^2 + r_{24}^2 - r_{12}^2 - r_{34}^2) + \text{ two similar terms} \\ & -(r_{12}^2 r_{13}^2 r_{23}^2 + \text{three similar terms}) \end{aligned} \quad (2.6)$$

We now state two formulas which will be used later:

1. Alternative form of A_4:

$$\begin{aligned} A_4 = \quad & \sum_{l=1}^{4} \big((d_3(r_{il}, r_{jl}, r_{kl}) + 8r_{il}r_{jl}r_{kl} + r_{il}(r_{il}^2 - r_{jk}^2) + \\ & r_{jl}(r_{jl}^2 - r_{ik}^2) + r_{kl}(r_{kl}^2 - r_{ij}^2)\big) \, d_3(r_{ij}, r_{ik}, r_{jk}), \end{aligned} \quad (2.7)$$

   where for each l we write $\{1, 2, 3, 4\} \setminus \{l\} = \{i < j < k\}$.

2. The sum of the second and the fourth term of (2.5) can be rewritten as

$$\begin{aligned} & 144V^2 - 2d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = \\ & \quad = (r_{12} - r_{34})^2 (r_{13}^2 r_{24}^2 + r_{14}^2 r_{23}^2 - r_{12}^2 r_{34}^2) + \text{ two such terms } + \\ & \quad + 4r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} - \\ & \quad - r_{12}^2 r_{13}^2 r_{23}^2 - r_{12}^2 r_{14}^2 r_{24}^2 - r_{13}^2 r_{14}^2 r_{34}^2 - r_{23}^2 r_{24}^2 r_{34}^2. \end{aligned} \quad (2.8)$$

It is well known that this quantity is always nonpositive.

The imaginary part $\Im(D_4))$ of Atiyah determinant can be written as a product of $144V^2$ with a polynomial (with integer coefficients) having 369 terms.

The normalized Atiyah determinant for 4 points is

$$\mathcal{D}_4 = \frac{D_4}{2^{\binom{4}{2}} \prod\limits_{1\leq i<j\leq 4} r_{ij}}.$$

The original Atiyah conjecture in our cases is equivalent to nonvanishing of the determinants D_3 and D_4.

A stronger conjecture of Atiyah and Sutcliffe ([4],Conjecture 2) states in our cases that $|D_3| \geq 8r_{12}r_{13}r_{23}$ ($\Leftrightarrow |\mathcal{D}_3| \geq 1$) and $|D_4| \geq 64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34}$ ($\Leftrightarrow |\mathcal{D}_4| \geq 1$).

From the formula (2.5) above, with the help of the simple inequality $d_3(a,b,c) \leq abc$ (for $a,b,c \geq 0$), Eastwood and Norbury got "almost" the proof of the stronger conjecture by exhibiting the inequality

$$\Re(D_4) \geq 60r_{12}r_{13}r_{23}r_{14}r_{24}r_{34}.$$

To remove the word "almost" seems to be not so easy (at present not yet done even for planar configuration of four points).

A third conjecture (stronger than the second) of Atiyah and Sutcliffe ([4], Conjecture 3) can be expressed, in the four point case, in terms of polynomials in the edge lengths as

$$|D_4|^2 \geq \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk}) \tag{2.9}$$

where the product runs over the four faces of the tetrahedron.
(cf. `ftp://ftp.maths.adelaide.edu.au/pure/meastwood/atiyah.ps`)

In the first part of this paper we study some infinite families of quadrilaterals and tetrahedra and verify both Atiyah and Sutcliffe conjectures for several such infinite families. In this version of the paper we propose a somewhat stronger conjecture than (2.9) which reads as follows:

Conjecture 2.1 *(Four Points Conjectures)*

$$\begin{aligned} &\Re(D_4) - (4+\tfrac{3}{4})\cdot 288V^2 \geq \\ &\geq 64 \prod_{1\leq i,j\leq 4} r_{ij} + \sum_{\{i<j<k\}\subset\{1,2,3,4\}} (4+\tfrac{1}{4}\delta) r_{il}r_{jl}r_{kl} d_3(r_{ij}, r_{ik}, r_{jk}) \end{aligned} \tag{2.10}$$

where

$$\delta = \begin{cases} \dfrac{d_3(r_{ij}, r_{ik}, r_{jk})}{r_{ij}r_{ik}r_{jk}} & , \textit{ weak version} \\ 1 & , \textit{ strong version} \end{cases}$$

Proposition 2.2 *Any of the Four Points Conjectures (2.10) imply conjecture (2.9).*

Proof .
By using the inequality $1 \geq d_3(a,b,c)/(abc), (a,b,c > 0)$ (see Appendix 2, Proposition 6.1) we see that the strong version implies the weak version of conjecture. We then rewrite the rhs of the weak version of (2.10) as follows:

$$\prod_{1\leq i,j\leq 4} r_{ij} \left(\frac{1}{4} \sum_{l=1}^{4} \left(\frac{d_3(r_{ij}, r_{ik}, r_{jk})}{r_{ij} r_{ik} r_{jk}} + 8 \right)^2 \right)$$

Finally, by the quadratic–geometric (QG) inequality we obtain

$$\geq \prod_{1\leq i,j\leq 4} r_{ij} \left(\prod_{l=1}^{4} \left(\frac{d_3(r_{ij}, r_{ik}, r_{jk})}{r_{ij} r_{ik} r_{jk}} + 8 \right) \right)^{\frac{2}{4}} = \left(\prod_{l=1}^{4} (d_3(r_{ij}, r_{ik}, r_{jk}) + 8 r_{ij} r_{ik} r_{jk}) \right)^{\frac{1}{2}}$$

Thus we obtain:

$$|D_4|^2 \geq |\Re(D_4)|^2 \geq |\Re(D_4) - (4+\tfrac{3}{4}) \cdot 288 V^2|^2 \geq \prod_{l=1}^{4} (d_3(r_{ij} r_{ik} r_{jk}) + 8 r_{ij} r_{ik} r_{jk})$$

i.e. the inequality (2.9). ■

Remark 2.3 *In terms of trigonometry (see subsection "Atiyah determinant for triangles and quadrilaterals via trigonometry" on page 20), the weak Four Points Conjecture can be written simply as*

$$\Re(D_4) - (4 + \tfrac{3}{4}) \cdot 288 V^2 \geq \left(\prod_{1\leq i,j\leq 4} r_{ij} \right) \left(4 \sum_{l=1}^{4} c_l^2 \right)$$

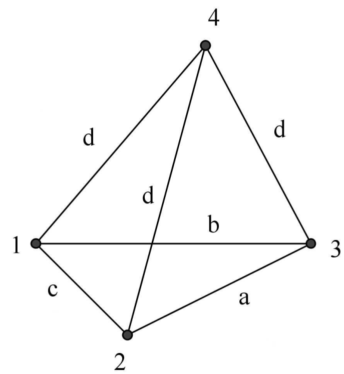
where

$$c_l := \cos^2 \frac{X^{(l)}}{2} + \cos^2 \frac{Y^{(l)}}{2} + \cos^2 \frac{Z^{(l)}}{2}, \quad l = 1, 2, 3, 4.$$

and $X^{(l)}$, $Y^{(l)}$, $Z^{(l)}$ are the angles of the triangle opposite to the vertex l.

2.1 Atiyah–Sutcliffe conjecture for (vertically) upright tetrahedra (or pyramids)

We call a tetrahedron upright if some of its vertices (say 4) is equidistant from all the remaining vertices (1, 2 and 3, which we can think as lying in a horizontal plane.)



$r_{23} = a$
$r_{13} = b$
$r_{12} = c$
$r_{14} = r_{24} = r_{34} = d$

Note that then $d \geq R =$ the circumradius of the base triangle 123, then by Heron's formula we have: $R = abc/\sqrt{(a+b+c)d_3(a,b,c)}$.

Here, as before, $d_3(a,b,c) = (a+b-c)(a-b+c)(-a+b+c)$, $(a,b,c>0)$.

The left hand side of the strong Four Points Conjecture 2.10 (but without $\frac{3}{4}$ term!) can be evaluated as follows, by using Eastwood-Norbury formula (2.5)

$$LHS = \Re(D_4) - 4 \cdot 288V^2 =$$
$$= 64 \prod_{1 \leq i < j \leq 4} r_{ij} - 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) + A_4 - 3 \cdot 288V^2$$

where

$$-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = -4d_3(a,b,c)d^3$$

$$A_4 = \sum_{l=1}^{4} \left(\sum_{(l \neq) i=1}^{4} r_{li}((r_{lj} + r_{lk})^2 - r_{jk}^2) d_3(r_{ij}, r_{ik}, r_{jk}) \right) =$$
$$= \sum_{cyc(a,b,c)} \left[c((b+d)^2 - d^2) + b((c+d)^2 - d^2) + d((b+c)^2 - a^2) \right] d_3(a,d,d) +$$
$$+ \left[d((d+d)^2 - a^2) + d((d+d)^2 - b^2) + d((d+d)^2 - c^2) \right] d_3(a,b,c) =$$
$$= [4bcd + ((b+c)^2 - a^2)d + b^2c + bc^2](2a^2d - a^3)] + \cdots +$$
$$+[12d^3 - (a^2+b^2+c^2)d]d_3(a,b,c) \qquad (by \;\; 2.7)$$

$$-3 \cdot 288V^2 =$$
$$= -6[(b^2+c^2-a^2)a^2d^2 + (c^2+a^2-b^2)b^2d^2 + (a^2+b^2-c^2)c^2d^2 - a^2b^2c^2](by \; 2.6)$$

Similarly the right hand side of the Conjecture 2.10

$$
\begin{aligned}
RHS = &\quad 64\prod_{1\le i<j\le 4} r_{ij} + \sum_{l=1}^{4}\left(4+\frac{1}{4}\right) r_{il}r_{jl}r_{kl}d_3(r_{ij},r_{ik},r_{jk}) \\
= &\quad 64abcd^3 + \left(4+\frac{1}{4}\right) bcd(2a^2d-a^3) + \text{ two such terms}+ \\
&\quad + \left(4+\frac{1}{4}\right) d^3 d_3(a,b,c)
\end{aligned}
$$

Now we can rewrite the difference

$$LHS - RHS = I + II$$

where

$$I = \sum_{cyc}(b^2c+bc^2)(2a^2d-a^3) - (a^2+b^2+c^2)d_3(a,b,c)d + 6a^2b^2c^2 - 24\frac{a^2b^2c^2}{a+b+c}d$$

and

$$
\begin{aligned}
II = &\quad \left(4-\frac{1}{4}\right) d_3(a,b,c)d^3 + \sum_{cyc}((b+c)^2-a^2)d(2a^2d-a^3)- \\
&\quad -6\sum_{cyc}(b^2+c^2-a^2)a^2d^2 - \frac{1}{4}\sum_{cyc} bcd(2a^2d-a^3) + 24\frac{a^2b^2c^2}{a+b+c}d
\end{aligned}
$$

Then we can further simplify

$$
\begin{aligned}
I = &\quad \left[4abc(ab+ac+bc) - (a^2+b^2+c^2)d_3(a,b,c) - \frac{24a^2b^2c^2}{a+b+c}\right] d+ \\
&\quad +6a^2b^2c^2 - \sum_{sym} a^3b^2c
\end{aligned}
$$

and

$$
\begin{aligned}
II = d\Bigg[&\frac{15}{4}d_3(a,b,c)d^2 + (a+b+c)\left(\frac{7}{2}abc - 4d_3(a,b,c)\right)d + 24\frac{a^2b^2c^2}{a+b+c}+ \\
&+\frac{1}{4}abc(a^2+b^2+c^2) - (a+b+c)\left(\sum_{sym} a^3b - a^4 - b^4 - c^4\right)\Bigg]
\end{aligned}
$$

Lemma 2.4 *We have the following strengthening of the basic inequality for our function $d_3(a,b,c) = (a+b-c)(a-b+c)(-a+b+c)$:*

$$d_3(a,b,c) \le \frac{9a^2b^2c^2}{(a+b+c)(a^2+b^2+c^2)} \quad \left(\le \frac{27a^2b^2c^2}{(a+b+c)^3} \le abc\right)$$

Proof .
We have

$$\begin{aligned}
&9a^2b^2c^2 - (a^2+b^2+c^2)(a+b+c)d_3(a,b,c) = \\
&= 9a^2b^2c^2 - (a^2+b^2+c^2)(2a^2b^2+2a^2c^2+2b^2c^2-a^4+b^4+c^4) = \\
&= 3a^2b^2c^2 - a^4b^2 - a^2b^4 - a^4c^2 - a^2c^4 - b^4c^2 - b^2c^4 + a^6 + b^6 + c^6 = \\
&(a^2-b^2)[a^2(a^2-c^2) - b^2(b^2-c^2)] + c^2(a^2-c^2)(b^2-c^2) \geq 0 \\
&\text{(if we assume } a \geq b \geq c \geq 0)
\end{aligned}$$

(a special instance of a Schur inequality) ■
(Note that this result follows from the formula $OG^2 = R^2 - (a^2+b^2+c^2)/9$ for the distance of the circumcenter and the centroid of a triangle.)

Now we have

Lemma 2.5 *The quantity I is increasing w.r.t. d and it is positive for $d \geq R$.*

Proof .
We prove that the coefficient of d in I is positive by using that $(ab+ac+bc)(a+b+c) \geq 9abc$ and Lemma 2.4 .

The proof of positivity of I reduces to the positivity of the following quantity:

$$\begin{aligned}
&\{[4abc(ab+ac+bc) - (a^2+b^2+c^2)d_3(a,b,c)](a+b+c) - 24a^2b^2c^2\}^2 - \\
&-(a+b+c)^3(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2-6abc)^2 d_3(a,b,c)
\end{aligned}$$

which by substituting $a = b+h$ and $b = c+k$ and then expanding has all coefficients positive (and ranging from 1 to 32151). ■

Lemma 2.6 *The quantity II is increasing w.r.t. d and it is positive for $d \geq R$.*

Proof .
Let $II = d \cdot III$. Then

$$\frac{\partial III}{\partial d} = \left(\tfrac{15}{2}d_3(a,b,c)d - \tfrac{a+b+c}{2}d_3(a,b,c)\right) + \tfrac{7(a+b+c)}{2}(abc - d_3(a,b,c))$$

The second term is positive by Proposition 6.1. For the first term we have:

$$\begin{aligned}
&\tfrac{15}{2}d_3(a,b,c)d - \tfrac{a+b+c}{2}d_3(a,b,c) \geq \tfrac{15}{2}d_3(a,b,c)R - \tfrac{a+b+c}{2}d_3(a,b,c) \geq \\
&\left(\tfrac{15abc}{(a+b+c)^{3/2}} - \sqrt{d_3(a,b,c)}\right)\tfrac{a+b+c}{2}\sqrt{d_3(a,b,c)} \geq 0
\end{aligned}$$

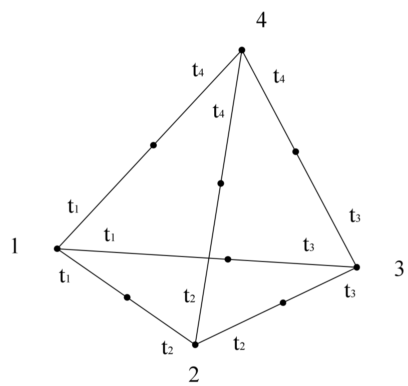
by Lemma (2.4).
The proof of positivity of II reduces to the positivity of the following quantity:

$$\begin{aligned}
&\frac{15}{4}d_3(a,b,c)R^2 + (a+b+c)\left(\frac{7}{2}abc - 4d_3(a,b,c)\right)R + 24\frac{a^2b^2c^2}{a+b+c} + \\
&+\tfrac{1}{4}abc(a^2+b^2+c^2) - (a+b+c)\left(\textstyle\sum_{sym} a^3b - a^4 - b^4 - c^4\right)
\end{aligned}$$

which can be nicely visualized by `Maple` using tangential coordinates ($a = v+w$, $b = u+w$, $c = u+v$). ■

2.2 Atiyah–Sutcliffe conjectures for edge–tangential tetrahedra

By edge–tangential tetrahedron we shall mean any tetrahedron for which there exists a sphere touching all its edges (i.e. its 1–skeleton has an inscribed sphere.) For each i from 1 to 4 we denote by t_i the length of the segment (lying on the tangent line) with one endpoint the vertex and the other the point of contact of the tangent line with a sphere.



$$r_{ij} = t_i + t_j, \ (1 \le i < j \le 4)$$

Now we shall compute all the ingredients appearing in the Eastwood–Norbury formula for D_4 in terms of elementary symmetric functions of the (tangential) variables t_1, t_2, t_3, t_4 (recall $e_1 = t_1 + t_2 + t_3 + t_4$, $e_2 = t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4$, $e_3 = t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4$, $e_4 = t_1t_2t_3t_4$).

$$\begin{aligned} 64 r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} &= 64 \prod_{1\le i<j\le 4}(t_i + t_j) = 64 s_{3,2,1} = \\ &= 64 \begin{vmatrix} e_3 & e_4 & 0 \\ e_1 & e_2 & e_3 \\ 0 & 1 & e_1 \end{vmatrix} = 64e_3e_2e_1 - 64e_4e_1^2 - 64e_3^2 \end{aligned}$$

Here we have used Jacobi–Trudi formula for the triangular Schur function $s_{3,2,1}$ (see [9], (3.5)). Furthermore we have

$$\begin{aligned} -4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) &= 128e_4e_2 - 32e_4e_1^2 - 32e_3^2 \\ 288V^2 &= 128e_4e_2 - 32e_3^2 \end{aligned}$$

In order to compute A_4 we first compute, for fixed l the following quantities

$$\begin{aligned} d_3(r_{ij}, r_{ik}, r_{jk}) &= 8t_it_jt_k \\ \sum_{(l\neq)i=1}^{4} r_{li}((r_{lj} + r_{lk})^2 - r_{jk}^2) &= 4(3t_l(t_1 + t_2 + t_3 + t_4) + 2(t_it_j + t_it_k + t_jt_k))t_l. \end{aligned}$$

Thus we get:

$$A_4 = 32(3e_1^2 + 4e_2)e_4 = 96e_4e_1^2 + 128e_4e_2.$$

Now we adjust terms in D_4, in order to get shorter expression, as follows

$$\begin{aligned}
D_4 = &\ (64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} - 2\cdot 288V^2)+\\
&+(-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) - 288V^2) + A_4 + 4\cdot 288V^2\\
= &\ (64e_3e_2e_1 - 64e_4e_1^2 - 256e_4e_2) + (-32e_4e_1^2)+\\
&+(96e_4e_1^2 + 128e_4e_2) + 4\cdot 288V^2\\
= &\ 64e_3e_2e_1 - 128e_4e_2 + 1152V^2\\
= &\ 64e_2(e_3e_1 - 2e_4) + 1152V^2\\
= &\ 64e_2(2e_4 + m_{211}) + 1152V^2,
\end{aligned}$$

where $m_{211} = t_1^2t_2t_3 + \cdots$ denotes the monomial symmetric function associated to the partition $(2,1,1)$.

In order to verify the third conjecture of Atiyah and Sutcliffe

$$|D_4|^2 \geq \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk})$$

we note first that

$$\begin{aligned}
d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk} &= (8t_it_jt_k + 8(t_i+t_j)(t_i+t_k)(t_j+t_k))\\
&= 8(t_i+t_j+t_k)(t_it_j + t_it_k + t_jt_k)
\end{aligned}$$

and state the following:

Lemma 2.7 *For any nonnegative real numbers $t_1, t_2, t_3, t_4 \geq 0$ the following inequality*

$$\begin{aligned}
&(t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4)^2(2t_1t_2t_3t_4 + m_{211}(t_1,t_2,t_3,t_4))^2 \geq\\
&\geq \prod\nolimits_{\{i<j<k\}\subset\{1,2,3,4\}}(t_i+t_j+t_k)(t_it_j+t_it_k+t_jt_k)
\end{aligned} \tag{2.11}$$

holds true.

Proof of Lemma 2.7.
The difference between the left hand side and the right hand side of the above inequality (2.11), written in terms of monomial symmetric functions is equal to

$$\begin{aligned}
LHS - RHS = &\ m_{6321} + 3m_{6222} + m_{543} + 2m_{5421} + 7m_{5322} + 5m_{5331}+\\
&+3m_{444} + 7m_{4431} + 8m_{4422} + 8m_{4332} + 3m_{3333} \geq 0
\end{aligned}$$

■

Remark 2.8 *One may think that the inequality in Lemma 2.7 can be obtained as a product of two simpler inequalities. This is not the case, because the following inequalities hold true:*

$$\begin{aligned}
(t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4)^2 &\leq \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (t_i+t_j+t_k)\\
(2t_1t_2t_3t_4 + m_{211}(t_1,t_2,t_3,t_4))^2 &\geq \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (t_it_j+t_it_k+t_jt_k)
\end{aligned}$$

Now we continue with verification of the third conjecture of Atiyah and Sutcliffe for edge tangential tetrahedron:

$$
\begin{aligned}
|D_4|^2 \geq \quad & (D_4)^2 \geq [64e_2(2e_4+m_{211})]^2 \\
\geq \quad & 8^4 \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (t_i+t_j+t_k)(t_it_j+t_it_k+t_jt_k) \quad \text{(by Lemma 2.7)} \\
= \quad & \prod_{\{i<j<k\}\subset\{1,2,3,4\}} (d_3(r_{ij},r_{ik},r_{jk})+8r_{ij}r_{ik}r_{jk})
\end{aligned}
$$

so the strongest Atiyah–Sutcliffe conjecture is verified for edge–tangential tetrahedra.

2.3 Verification of the strong Four Points Conjecture for edge–tangential tetrahedra

The strong Four Points Conjecture 2.10 for edge tangential tetrahedra is equivalent to positivity of the following quantity:

$$
\begin{aligned}
& \Re(D_4) - 64\prod r_{ij} - (4+\tfrac{3}{4})288V^2 - \sum_{l=1}^{4}(4+\tfrac{1}{4})r_{il}r_{jl}r_{kl}\, d_3(r_{ij},r_{ik},r_{jk}) \\
& = (-d_3(r_{12}r_{34},r_{13}r_{24},r_{14}r_{23})+A_4) + 288V^2 - (4+\tfrac{3}{4})288V^2 \\
& \quad - \sum_{l=1}^{4}(4+\tfrac{1}{4})r_{il}r_{jl}r_{kl}\, d_3(r_{ij},r_{ik},r_{jk}) \\
& = (-32m_{3111} - 32m_{222} + 96m_{3111} + 320m_{2211}) - 240m_{2211} + 120m_{222} \\
& \quad - (34m_{3111} + 136m_{2211} + 34m_{222}) \\
& = 30m_{3111} + 54m_{222} - 56m_{2211}
\end{aligned}
$$

In terms of augmented monomial symmetric functions

$$
\widetilde{m}_\lambda(t_1,t_2,t_3,t_4) = \sum_{\sigma\in S_4} t^{\sigma.\lambda}
$$

the last quantity is equal to

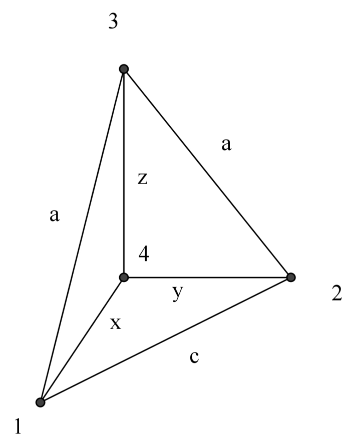
$$
= 5\widetilde{m}_{3111} + 9\widetilde{m}_{222} - 14\widetilde{m}_{2211} \ (\geq 0 \text{ by Muirheads's inequality})
$$

Thus, the strong Four Points Conjecture is verified for the edge–tangential tetrahedra.

Note that the verification of this conjecture which is stronger than A–S conjecture C3 is somewhat simpler (at least for edge–tangential tetrahedra).

2.4 Trirectangular tetrahedra

A tetrahedron is called trirectangular if it has a vertex at which all the face angles are right angles. The opposite face to such a vertex we call a base. We label the edge lengths as follows



$$\begin{aligned} r_{12} &= c \\ r_{13} &= b \\ r_{23} &= a \\ r_{14} &= x \\ r_{24} &= y \\ r_{34} &= z \end{aligned}$$

We have following obvious relations: $a^2 = y^2 + z^2$, $b^2 = x^2 + z^2$, $c^2 = x^2 + y^2$. By using them we can get

$$\begin{aligned} d_3(a,b,c) &= 2(ax^2 + by^2 + cz^2 - abc), \\ d_3(x,y,c) &= 2xy(x+y-c), \\ d_3(x,b,z) &= 2xz(x+z-b), \\ d_3(a,y,z) &= 2yz(y+z-a) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &\Re(D_4) - 64abcxyz - 288V^2 = \\ &= 4xyz\sum_{cyc} 2ax^2 + \sum_{cyc}(2ab + cz + z^2)(x+y) - 10abc \end{aligned} \tag{2.13}$$

where $\sum_{cyc}$ has three terms [1] corresponding to a cycle $((a,x) \to (b,y) \to (c,z))$.

By writing $x + y = x + y - c + c$ and using the identity

$$\sum_{cyc} c^2 z = \sum_{cyc}(x^2+y^2)z = \sum_{cyc}(x+y)z^2 = \sum_{cyc} z^2(x+y-c) + \sum_{cyc} ax^2$$

we get that the second cyclic sum is equal to

$$\sum_{cyc}(2ab+cz+z^2)(x+y) = 6ab + \sum_{cyc}(2ab+cz+2z^2)(x+y-c) + 2\sum_{cyc} ax^2 \tag{2.14}$$

By inserting this into (2.13) we get

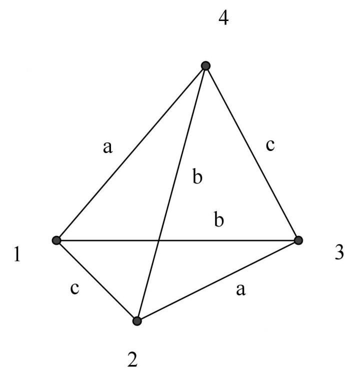
$$\Re(D_4) - 64abcxyz - 288V^2 = 4xyz(2d_3(a,b,c) + \sum_{cyc}(2ab+cz+2z^2)(x+y-c))$$

Hence $\Re(D_4) \geq 64abcxyz$ so the verification of the C2 of Atiyah–Sutcliffe for trirectangular tetrahedra is finished.

[1] $\sum_{cyc} f(a,b,c,x,y,z) = f(a,b,c,x,y,z) + f(b,c,a,y,z,x) + f(c,a,b,z,x,y)$

2.5 Atiyah–Sutcliffe conjectures for regular and semi–regular tetrahedra

Semiregular (SR) tetrahedra are one of the simplest configurations of tetrahedra. These tetrahedra have opposite edges equal and hence all faces are congruent. Sometimes semi–regular tetrahedra are called isosceles tetrahedra.



$$r_{23} = r_{14} = a$$
$$r_{13} = r_{24} = b$$
$$r_{12} = r_{34} = c$$

By (2.8) we get

$$288V^2 - 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = 0 \; (\Rightarrow 288V^2 = 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}))$$

By (2.7) we get

$$\begin{aligned} A_4 &= \sum_{l=1}^{4} (d_3(r_{il}, r_{jl}, r_{kl}) + 8r_{il}r_{jl}r_{kl}) d_3(r_{ij}, r_{ik}, r_{jk}) \\ &= 4d_3(a,b,c)^2 + 32abc\, d_3(a,b,c) \end{aligned}$$

The quantity in the weak Four Points Conjecture is

$$\begin{aligned} & l.h.s - r.h.s = \\ &= A_4 - \left(4 + \frac{3}{4}\right) 288V^2 - \sum_{l=1}^{4} \left(4 + \frac{1}{4} \frac{d_3(r_{ij}, r_{ik}, r_{jk})}{r_{il}r_{jl}r_{kl}}\right) r_{ij}r_{ik}r_{jk} d_3(r_{ij}, r_{ik}, r_{jk}) \\ &= 4d_3(a,b,c)^2 + 32abc\, d_3(a,b,c) - (16+3)d_3(a^2,b^2,c^2) - [16abc\, d_3(a,b,c) + d_3(a,b,c)^2] \\ &= 3(d_3(a,b,c)^2 - d_3(a^2,b^2,c^2)) + 16(abc\, d_3(a,b,c) - d_3(a^2,b^2,c^2)) \geq 0 \end{aligned}$$

by using the inequalities $abc \geq d_3(a,b,c)$ and $d_3(a,b,c)^2 \geq d_3(a^2,b^2,c^2)$ (see Appendix 2, Proposition 6.1; also see [12] or [13]).

This proves the weak Four Points Conjecture for semiregular tetrahedra. ■

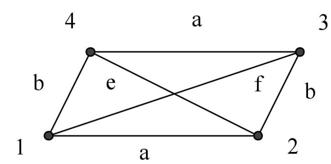
The proof of the strong Four Points Conjecture for semiregular tetrahedra reduces to the positivity of the following expression

$$4(d_3(a,b,c)^2 - d_3(a^2,b^2,c^2)) + 15(abc\, d_3(a,b,c) - d_3(a^2,b^2,c^2)) \geq 0$$

which is also true by the same argument.

2.6 Atiyah–Sutcliffe conjectures for parallelograms

Given a parallelogram with vertices 1, 2, 3 and 4 denote by a, b its side lengths and by e, f its diagonals.



$r_{12} = r_{34} = a$
$r_{23} = r_{14} = b$
$r_{24} = e$
$r_{13} = f$

For the numbers a, b, e, f we have the basic relation ("a parallelogram law")

$$e^2 + f^2 = 2(a^2 + b^2) \tag{2.15}$$

By using this relation we can rewrite various quantities in the Eastwood-Norbury formula.

Proposition 2.9 *We have the following identities*

1. $d_3(a, b, e) = (a+b-e)(a-b+e)(-a+b+e) = (a+b-e)(a+b-f)(a+b+f)$
2. $\Delta := (a + b + e)d_3(a, b, e) = (a + b + f)d_3(a, b, f) =$
   $= (a + b + e)(a + b + f)(a + b - e)(a + b - f) =$
   $= 2a^2b^2 + 2a^2e^2 + 2b^2e^2 - a^4 - b^4 - e^4 = 2a^2b^2 + 2a^2f^2 + 2b^2f^2 - a^4 - b^4 - f^4$
3. $4ab + e^2 - f^2 = 2(a+b+f)(a+b-f)$, $4ab + f^2 - e^2 = 2(a+b+e)(a+b-e)$
4. $d_3(a^2, b^2, ef) = (a^2 + b^2 - ef)\Delta$
5. $d_3(a, b, e)d_3(a, b, f) - d_3(a^2, b^2, ef) = (2ab - 2ef - (a + b)(e + f))\Delta$
6. $ed_3(a, b, f) + fd_3(a, b, e) = (a + b - e)(a + b - f)(e^2 + f^2 + (a + b)(e + f))$
7. $(4ab + e^2 - f^2)ed_3(a, b, f) + (4ab + f^2 - e^2)fd_3(a, b, e) = 2((a + b)(e + f) - 2ef)\Delta$

Proof .
For *1.* we write $(a-b+e)(-a+b+e) = e^2-(a-b)^2 = 2a^2+2b^2-f^2-(a-b)^2 = (a+b-f)(a+b+f)$. Identity *2.* follows from *1.* directly. For *3.* we substitute $e^2 = 2a^2 + 2b^2 - f^2$ and simplify. For *4.* we compute and use *2.*:

$$(a^2-b^2+ef)(-a^2+b^2+ef) = e^2f^2-(a^2-b^2)^2 = e^2(2a^2+2b^2-e^2)+2a^2b^2-a^4-b^4 = \Delta$$

For *5.* we first use *1.* and then *4.*: $d_3(a, b, e)d_3(a, b, f) - d_3(a^2, b^2, ef) = (a+b+f)(a+b-e)(a+b-f)d_3(a, b, f) - (a^2+b^2-ef)\Delta = [(a+b)^2-(a+b)(e+f)+ef]\Delta - (a^2 + b^2 - ef)\Delta = [2ab + 2ef - (a + b)(e + f)]\Delta$

For *6.* we use *1.* twice.
For *7.* we first use *3.* and then *2.*:

$$\begin{aligned} l.h.s. &= 2(a+b+f)(a+b-f)ed_3(a,b,f) + 2(a+b+e)(a+b-e)fd_(a,b,e) \\ &= 2[(a+b-f)e + (a+b-e)f]\Delta \end{aligned}$$

■

Now we apply Eastwood-Norbury formula (note that $288V^2 = 0$, $D_4 = real$)

$$D_4 - 64\prod r_{ij} = -4d_3(a^2,b^2,c^2) + A_4$$

where

$$\begin{aligned} A_4 &= 2[d_3(a,b,e) + 8abc + e(e^2-f^2)]d_3(a,b,f) + 2[d_3(a,b,f) + 8abc + f(f^2-e^2)]d_3(a,b,e) \\ &= I_0 + I_1 + I_2 \end{aligned}$$

where

$$\begin{aligned} I_0 &= 4d_3(a,b,e)d_3(a,b,f) \\ I_1 &= 2[4abe + e(e^2-f^2)]d_3(a,b,f) + 2[4abf + f(f^2-e^2)]d_3(a,b,e) \\ &= 4((a+b)(e+f) - 2ef)\Delta \quad \text{(by 7.)} \\ I_2 &= 2[4abe\, d_3(a,b,f) + 4abf\, d_3(a,b,e)] \\ &= 8ab(a+b-e)(a+b-f)(e^2+f^2+(a+b)(e+f)) \quad \text{(by 6.)} \end{aligned}$$

By using *5.* we have

$$\begin{aligned} D_4 - 64\prod r_{ij} &= 4(d_3(a,b,e)d_3(a,b,f) - d_3(a^2,b^2,ef)) + I_1 + I_2 \\ &= 4((2ab+2ef-(a+b)(e+f))\Delta + ((a+b)(e+f)-2ef)\Delta) + I_2 \\ &= 8ab\Delta + I_2 \geq 0 \end{aligned}$$

This proves the Atiyah–Sutcliffe conjecture (C2) for parallelograms. The Atiyah–Sutcliffe conjecture (C3) for parallelograms

$$D_4^2 \geq (d_3(a,b,e) + 8abe)^2(d_3(a,b,f) + 8abf)^2$$

is equivalent to the positivity of

$$D_4 - d_3(a,b,e)d_3(a,b,f) - 8[abf\, d_3(a,b,e) + abe\, d_3(a,b,f)] - 64a^2b^2ef \geq 0$$

but we can prove even stronger statement

$$\begin{aligned} &D_4 - 2d_3(a,b,e)d_3(a,b,f) - 8[abf\, d_3(a,b,e) + abe\, d_3(a,b,f)] - 64a^2b^2ef = \\ &= 8ab\Delta - 2d_3(a,b,e)d_3(a,b,f) = 2[4ab - (a+b-e)(a+b-f)]\Delta \geq 0 \end{aligned}$$

because the triangle inequalities $b < e + a$ and $a < f + b$ imply

$$(a+b-e)(a+b-f) < 2a \cdot 2b = 4ab.$$

Thus we have verified also C3 for parallelograms.

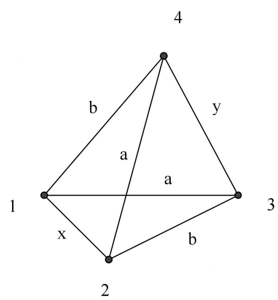
Finally we verify our strong Four Point Conjecture for parallelograms as follows

$$
\begin{aligned}
&D_4 - 64\prod r_{ij} - \sum(4+\tfrac{1}{4}) r_{il} r_{jl} r_{kl} d_3(r_{ij}, r_{jl}, r_{ik})\\
&= 8ab\Delta - \tfrac{1}{4}(I_2/4)\\
&= 8ab(a+b-e)(a+b-f)[(a+b+e)(a+b+f) - \tfrac{1}{16}(e^2+f^2+(a+b)(e+f))]\\
&= \tfrac{1}{2}ab(a+b-e)(a+b-f)[16((a+b)^2+(a+b)(e+f)+ef) - (2a^2+2b^2+(a+b)(e+f))]\\
&= \tfrac{1}{2}ab(a+b-e)(a+b-f)[14(a^2+b^2)+32ab+15(a+b)(e+f)+16ef] \geq 0
\end{aligned}
$$

■

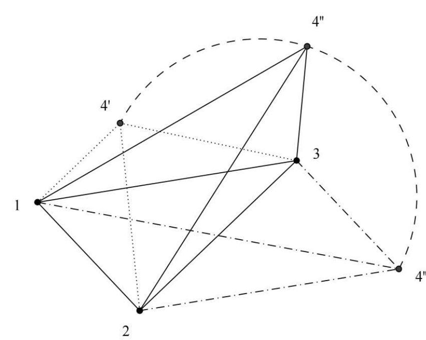
2.7 Atiyah–Sutcliffe conjectures for "wedge" tetrahedra

A tetrahedron with two pairs of opposite edges having the same length we simply call a "wedge" tetrahedron.



$r_{12} = x$
$r_{34} = y$
$r_{13} = r_{24} = a$
$r_{23} = r_{14} = b$

If $x = y = c$ we get a semiregular tetrahedron and if all points lie in a plane then we get either a parallelogram or an isosceles trapezium.



Again we compute the data appearing in the Eastwood–Norbury formula

$$
\begin{aligned}
&-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) =\\
&= -4d_3(xy, a^2, b^2)\\
&= -4(xy - a^2 + b^2)(xy + a^2 - b^2)(a^2 + b^2 - xy)
\end{aligned}
\tag{2.16}
$$

and we have the basic inequalities

$$
xy + b^2 \geq a^2,\ xy + a^2 \geq b^2,\ a^2 + b^2 \geq xy \tag{2.17}
$$

The positivity of the volume

$$144V^2 = xy - a^2 + b^2)(xy + a^2 - b^2)(2a^2 + 2b^2 - x^2 - y^2) \tag{2.18}$$

gives us one more basic inequality

$$2a^2 + 2b^2 \geq x^2 + y^2 \tag{2.19}$$

We have

$$\begin{aligned} A_4 = &\, 2[a((b+x)^2 - a^2) + b((a+x)^2 - b^2) + x((a+b)^2 - y^2)]d_3(a,b,y) \\ &+ 2[a((b+y)^2 - a^2) + b((a+y)^2 - b^2) + y((a+b)^2 - x^2)]d_3(a,b,x) \end{aligned} \tag{2.20}$$

By using identity

$$d_3(a,b,c) = a(b^2 + c^2 - a^2) + b(a^2 + c^2 - b^2) + c(a^2 + b^2 - c^2) - 2abc \tag{2.21}$$

we can rewrite A_4 as follows

$$\begin{aligned} A_4 &= 2[4abx + d_3(a,b,x) - x(a^2 + b^2 - x^2) + 2abx + x((a+b)^2 - y^2)]d_3(a,b,y) \\ &\quad + 2[4aby + d_3(a,b,y) - y(a^2 + b^2 - y^2) + 2aby + y((a+b)^2 - x^2)]d_3(a,b,x) \\ &= 2[4abx + d_3(a,b,x) - x((a-b)^2 - x^2) + x((a+b)^2 - y^2)]d_3(a,b,y) \\ &\quad + 2[4aby + d_3(a,b,y) - y((a-b)^2 - y^2) + y((a+b)^2 - x^2)]d_3(a,b,x) \\ &= \{8abx + d_3(a,b,x) + [d_3(a,b,x) - 2x((a-b)^2 - x^2)] + 2x((a+b)^2 - y^2)\}d_3(a,b,y) \\ &\quad + \{8aby + d_3(a,b,y) + [d_3(a,b,y) - 2y((a-b)^2 - y^2)] + 2y((a+b)^2 - x^2)\}d_3(a,b,x) \end{aligned} \tag{2.22}$$

Now we compute

$$\begin{aligned} & d_3(a,b,x) - 2x((a-b)^2 - x^2) = \\ &= (a+b-x)(a-b+x)(-a+b+x) + 2x(a-b+x)(-a+b+x) \\ &= (a+b+x)(a-b+x)(-a+b+x) \end{aligned}$$

The contribution $A_4^{[\,]}$ of both square brackets in A_4 is equal to

$$\begin{aligned} A_4^{[\,]} &:= [d_3(a,b,x) - 2x((a-b)^2 - x^2)]d_3(a,b,y) + \\ &\quad + [d_3(a,b,y) - 2y((a-b)^2 - y^2)]d_3(a,b,x) \\ &= (x^2 - (a-b)^2)(y^2 - (a-b)^2)[(a+b+x)(a+b-y) + (a+b+y)(a+b-x)] \\ &= (x^2 - (a-b)^2)(y^2 - (a-b)^2)(2(a+b)^2 - 2xy) \\ &= 4ab(x^2 - (a-b)^2)(y^2 - (a-b)^2) + 2(x^2 - (a-b)^2)(y^2 - (a-b)^2)(a^2 + b^2 - xy) \end{aligned} \tag{2.23}$$

At this point we have discovered the following beautiful identity

$$\begin{aligned} & [x^2 - (a-b)^2][y^2 - (a-b)^2] = \\ &= (xy - a^2 + b^2)(xy + a^2 - b^2) + (a-b)^2(2a^2 + 2b^2 - x^2 - y^2) \end{aligned} \tag{2.24}$$

By this identity we can write

$$\begin{aligned} A_4^{[\,]} = & \, 4ab(x^2-(a-b)^2)(y^2-(a-b)^2) \\ & + 2(a-b)^2(2a^2+2b^2-x^2-y^2)(a^2+b^2-xy) \\ & + 2d_3(a^2,b^2,xy) \end{aligned}$$

Lemma 2.10 *We have the following inequality for "wedge" tetrahedra*

$$d_3(a^2,b^2,xy) \le 2ab(x^2-(a-b)^2)(y^2-(a-b)^2)$$

Proof .
Recall that

$$d_3(a^2,b^2,xy) = (a^2+b^2-xy)(a^2-b^2+xy)(-a^2+b^2+xy)$$

Let $a \ge b$. Then the triangle inequalities $a \le b+x$ and $a \le b+y$ imply $(a-b)^2 \le xy$ i.e. $a^2+b^2-xy \le 2ab$. Since $2a^2+2b^2-x^2-y^2 \ge 0$ (inequality (2.19)) then from our inequality (2.24) it follows that

$$(a^2-2b^2+xy)(-a^2+b^2+xy) \le (x^2-(a-b)^2)(y^2-(a-b)^2)$$

By multiplying the last two inequalities Lemma follows. ■

As a consequence of Lemma we get immediately that

$$A_4 \ge A_4^{[\,]} \ge 4d_3(a^2,b^2,xy)$$

because the remaining terms in A_4 are all nonnegative. This verifies the A–S conjecture C2 for "wedge" tetrahedra.

Remark 2.11 *Instead of splitting* $2(a+b)^2-2xy = 4ab+2(a^2+b^2-xy)$ *(used above), we can use the identity*

$$2(a+b)^2-2xy = 4(a^2+b^2-xy)+2(xy-(a-b)^2)$$

to obtain explicit formula for $A_4^{[\,]}$*:*

$$\begin{aligned} & A_4^{[\,]} = \\ & [4(a^2+b^2-xy)+2(xy-(a-b)^2)][(xy-a^2+b^2)(xy+a^2-b^2)+ \\ & +(a-b)^2(2a^2+2b^2-x^2-y^2)] = \\ & = 4d_3(a^2,b^2,xy)+4(a^2+b^2-xy)(2a^2+2b^2-x^2-y^2)(a-b)^2+ \\ & +2(xy-(a-b)^2)(x^2-(a-b)^2)(y^2-(a-b)^2) \end{aligned}$$

which, without using Lemma 2.10, implies inequality

$$A_4^{[\,]} \ge 4d_3(a^2,b^2,xy)$$

needed for the verification of A–S conjecture C2 for "wedge" tetrahedra.

Now we state a final formula for "wedge" tetrahedra:

$$\boxed{\begin{array}{rl} & \underline{\text{First explicit formula for wedge tetrahedra}}: \\ \Re(D_4) = & (d_3(a,b,x)+8abx)(d_3(a,b,y)+8aby)+d_3(a,b,x)d_3(a,b,y)+ \\ & +2x((a+b)^2-y^2)d_3(a,b,y)+2y((a+b)^2-x^2)d_3(a,b,x)+ \\ & +4(a^2+b^2-xy)(2a^2+2b^2-x^2-y^2)(a-b)^2+ \\ & +2(xy-(a-b)^2)(x^2-(a-b)^2)(y^2-(a-b)^2) \\ & +288V^2 \end{array}}$$

which implies a strengthened A–S conjecture C3 for wedge tetrahedra

$$\begin{array}{rl} \Re(D_4) \geq & (d_3(a,b,x)+8abx)(d_3(a,b,y)+8aby)+d_3(a,b,x)d_3(a,b,y)+288V^2 \\ \geq & (d_3(a,b,x)+8abx)(d_3(a,b,y)+8aby) \end{array}$$

In the sequel we obtain an alternative formula for the real part of the Atiyah determinant for a wedge tetrahedra.

We group terms in A_4 differently as follows:

$$\begin{array}{rl} A_4 = & 2[4abx+d_3(a,b,x)+x(4ab+x^2-y^2)]d_3(a,b,y)+ \\ & +2[4aby+d_3(a,b,y)+x(4ab+y^2-x^2)]d_3(a,b,x) \end{array}$$

By letting

$$2s^2+2b^2-x^2-y^2 =: 2h \quad (\geq 0)$$

we can rewrite

$$4ab+x^2-y^2 = 4ab+x^2+(2h+x^2-2a^2-2b^2) = 2(h+x^2-(a-b)^2)$$

and similarly for

$$4ab+y^2-x^2 = 2(h+y^2-(a-b)^2)$$

Thus

$$\begin{array}{rl} A_4 = & 4d_3(a,b,x)d_3(a,b,y)+8abx\,d_3(a,b,y)+8aby\,d_3(a,b,x)+ \\ & +4h(x\,d_3(a,b,y)+y\,d_3(a,b,x))+4A_4' \end{array}$$

where

$$\begin{array}{rl} A_4' & = x(x^2-(a-b)^2)d_3(a,b,y)+y(y^2-(a-b)^2)d_3(a,b,x) \\ & = (x^2-(a-b)^2)(y^2-(a-b)^2)[x(a+b-y)+y(a+b-x)] \\ & = (x^2-(a-b)^2)(y^2-(a-b)^2)[(x-y)^2+x(a+b-x)+y(a+b-y)] \\ & = [(xy-a^2+b^2)(xy+a^2-b^2)+2(a-b)^2h][(x-y)^2+x(a+b-x)+y(a+b-y)] \end{array} \tag{2.25}$$

by our identity (2.24).

Note that

$$-144V^2+2d_3(a^2,b^2,xy) = (xy-a^2+b^2)(xy+a^2-b^2)(x-y)^2$$

So

$$
\begin{aligned}
A_4' = {} & (2d_3(a^2,b^2,xy) - 144V^2) + 2(a-b)^2 h[x(a+b-y) + y(a+b-x)] \\
& + (xy - a^2 + b^2)(xy + a^2 - b^2)(x(a+b-x) + y(a+b-y))
\end{aligned}
$$

By writing

$$
\begin{aligned}
& 4A_4' = 2A_4' + 2A_4' = \\
& = 2(x^2 - (a-b)^2)(y^2 - (a-b)^2)[x(a+b-y) + y(a+b-x)] + \\
& + \{4d_3(a^2,b^2,xy) - 288V^2 + 4(a-b)^2 h[x(a+b-y) + y(a+b-x)] + \\
& + 2(xy - a^2 + b^2)(xy + a^2 - b^2)(x(a+b-x) + y(a+b-y))\} = \\
& = 4d_3(a^2,b^2,xy) - 288V^2 + [2(x^2 - (a-b)^2)(y^2 - (a-b)^2) + 4(a-b)^2 h]\cdot \\
& \cdot (x(a+b-y) + y(a+b-x)) + \\
& + 2(xy - a^2 + b^2)(xy + a^2 - b^2)[x(a+b-x) + y(a+b-y)]
\end{aligned}
$$

we obtain the following explicit formula for the real part of Atiyah determinant for "wedge" tetrahedron:

$$
\boxed{
\begin{aligned}
& \underline{\text{Second explicit formula for wedge tetrahedra}}\text{:} \\
\Re(D_4) = {} & (d_3(a,b,x) + 8abx)(d_3(a,b,y) + 8aby) + 3d_3(a,b,x)d_3(a,b,y) + \\
& + 2x((a+b)^2 - y^2)d_3(a,b,y) + 2y((a+b)^2 - x^2)d_3(a,b,x) + \\
& 2(x^2y^2 - (a^2 - b^2))[x(a+b-x) + y(a+b-y)] + \\
& + 2[(a-b)^2(x(a+b-y) + y(a+b-x))](2a^2 + 2b^2 - x^2 - y^2)
\end{aligned}
}
$$

which implies another strengthening of the Atiyah–Sutcliffe conjecture C3 for "wedge" tetrahedra

$$
\begin{aligned}
\Re(D_4) & \geq (d_3(a,b,x) + 8abx)(d_3(a,b,y) + 8aby) + 3d_3(a,b,x)d_3(a,b,y) \\
& \geq (d_3(a,b,x) + 8abx)(d_3(a,b,y) + 8aby)
\end{aligned}
$$

2.8 Atiyah determinant for triangles and quadrilaterals via trigonometry

Denote the three points x_1, x_2, x_3 simply by symbols $1, 2, 3$ and let X, Y and Z denote the angles of the triangle at vertices 1, 2 and 3 respectively. Then we can express the Atiyah determinant $D_3 = d_3(r_{12}, r_{13}, r_{23}) + 8r_{12}r_{13}r_{23}$ as follows

$$
D_3 = 4r_{12}r_{13}r_{23}\left(\cos^2\frac{X}{2} + \cos^2\frac{Y}{2} + \cos^2\frac{Z}{2}\right).
$$

This follows, by using cosine law and sum to product formula for cosine, from the following identity

$$
\begin{aligned}
d_3(a,b,c) + 8abc & = (a+b-c)(a-b+c)(-a+b+c) + 8abc \\
& = a((b+c)^2 - a^2) + b((c+a)^2 - b^2) + c((a+b)^2 - c^2).
\end{aligned}
$$

Now we shall translate the Eastwood–Norbury formula for (planar quadrilaterals) into a trigonometric form. Denote the four points x_1, x_2, x_3, x_4 simply by symbols $1, 2, 3, 4$ and denote by

$$
(X^{(1)}, Y^{(1)}, Z^{(1)}),\quad (X^{(2)}, Y^{(2)}, Z^{(2)}),\quad (X^{(3)}, Y^{(3)}, Z^{(3)}),\quad (X^{(4)}, Y^{(4)}, Z^{(4)})
$$

the angles of the triangles 234, 341, 412, 123 in this cyclic order (i.e. the angle of a triangle 412 at vertex 2 is $Z^{(3)}$ etc.).

Next we denote by c_l, $(1 \le l \le 4)$, the sums of cosines squared of half-angles of the l–th triangle i.e.:

$$c_l := \cos^2 \frac{X^{(l)}}{2} + \cos^2 \frac{Y^{(l)}}{2} + \cos^2 \frac{Z^{(l)}}{2}, \quad l = 1, 2, 3, 4.$$

Similarly, we denote by $\widehat{c}_l$, $(1 \le l \le 4)$, the sum of cosines squared of half-angles at the l–th vertex of our quadrilateral thus

$$\begin{aligned}
\widehat{c}_1 &= \cos^2 \frac{Z^{(2)}}{2} + \cos^2 \frac{Y^{(3)}}{2} + \cos^2 \frac{X^{(4)}}{2} \\
\widehat{c}_2 &= \cos^2 \frac{Z^{(3)}}{2} + \cos^2 \frac{Y^{(4)}}{2} + \cos^2 \frac{X^{(1)}}{2} \\
\widehat{c}_3 &= \cos^2 \frac{Z^{(4)}}{2} + \cos^2 \frac{Y^{(1)}}{2} + \cos^2 \frac{X^{(2)}}{2} \\
\widehat{c}_4 &= \cos^2 \frac{Z^{(1)}}{2} + \cos^2 \frac{Y^{(2)}}{2} + \cos^2 \frac{X^{(3)}}{2}
\end{aligned}$$

Then the term A_4 in the Eastwood–Norbury formula can be rewritten as

$$\begin{aligned}
A_4 &= \sum_{l=1}^{4} (4 r_{li} r_{lj} r_{lk} \widehat{c}_l) \cdot 4 r_{ij} r_{ik} r_{jk} (c_l - 2) \\
&= 16 r_{12} r_{13} r_{23} r_{14} r_{24} r_{34} \sum_{l=1}^{4} \widehat{c}_l (c_l - 2).
\end{aligned}$$

where for each l we write $\{1, 2, 3, 4\} \setminus \{l\} = \{i < j < k\}$.

In order to rewrite the term $-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23})$ into a trigonometric form we recall a theorem of Möbius ([10]) which claims that for any quadrilateral 1234 in a plane the products $r_{12}r_{34}$, $r_{13}r_{24}$ and $r_{14}r_{23}$ are proportional to the sides of a triangle whose angles are the differences of angles in the quadrilateral 1234:

$$\begin{aligned}
X &= \sphericalangle 134 - \sphericalangle 124 \\
Y &= \sphericalangle 214 - \sphericalangle 234 \\
Z &= \sphericalangle 413 - \sphericalangle 423
\end{aligned}$$

Thus

$$-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = -16 r_{12} r_{13} r_{23} r_{14} r_{24} r_{34} (c - 2)$$

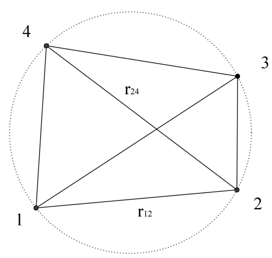
where

$$c = \cos^2 \frac{X}{2} + \cos^2 \frac{Y}{2} + \cos^2 \frac{Z}{2}.$$

Thus we have obtained a trigonometric formula for Atiyah determinant of quadrilaterals

$$\begin{aligned}\Re(D_4) &= \prod_{1\le i<j\le 4} r_{ij}\left(64-16(c-2)+16\sum_{l=1}^{4}\widehat{c}_l(c_l-2)\right)\\ &= 16\prod_{1\le i<j\le 4} r_{ij}\left(6-c+\sum_{l=1}^{4}\widehat{c}_l(c_l-2)\right)\end{aligned}$$

Now we shall verify Atiyah–Sutcliffe conjecture for cyclic quadrilaterals.



Ptolemy's theorem

$$r_{12}r_{34}+r_{23}r_{14}=r_{13}r_{24}$$

In this case, by a well known Ptolemy's theorem, we see that

$$-4d_3(r_{12}r_{34},r_{13}r_{24},r_{14}r_{23})=0 \quad (\Leftrightarrow c=2)$$

By using the equality of angles $Z^{(2)}=X^{(1)}$, $Z^{(3)}=X^{(2)}$, $Z^{(4)}=X^{(3)}$, $Z^{(1)}=X^{(4)}$ and $Y^{(1)}+Y^{(3)}=\pi=Y^{(2)}+Y^{(4)}$ (angles with vertex on a circle's circumference with the same endpoints are equal or supplement of each other)we obtain

$$\begin{aligned}\widehat{c}_1 &= \cos^2\frac{X^{(1)}}{2}+\sin^2\frac{Y^{(1)}}{2}+\cos^2\frac{Z^{(1)}}{2}=c_1-\cos Y^{(1)},\\ \widehat{c}_2 &= \cos^2\frac{X^{(2)}}{2}+\sin^2\frac{Y^{(2)}}{2}+\cos^2\frac{Z^{(2)}}{2}=c_2-\cos Y^{(2)},\\ \widehat{c}_3 &= \cos^2\frac{X^{(3)}}{2}+\sin^2\frac{Y^{(3)}}{2}+\cos^2\frac{Z^{(3)}}{2}=c_3-\cos Y^{(3)},\\ \widehat{c}_4 &= \cos^2\frac{X^{(4)}}{2}+\sin^2\frac{Y^{(4)}}{2}+\cos^2\frac{Z^{(4)}}{2}=c_4-\cos Y^{(4)}.\end{aligned}$$

Now we have

$$\begin{aligned}\Re(D_4) = &\left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(64+16\sum_{l=1}^{4}(c_l-\cos Y^{(l)})(c_l-2)\right)\\ \ge &\left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(64+16\sum_{l=1}^{4}(c_l-1)(c_l-2)\right)\end{aligned}$$

(here we have used that $2 \le c_l (\le \frac{9}{4})$ for each $l = 1,2,3,4$)

$$
\begin{aligned}
&\ge \left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(64 + 16\sum_{l=1}^{4}(c_l-2) + 16\sum_{l=1}^{4}(c_l-2)^2\right)\\
&\ge \left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(64 + 16\sum_{l=1}^{4}(c_l-2) + 4\left(\sum_{l=1}^{4}(c_l-2)\right)^2\right)
\end{aligned}
$$

(by quadratic–arithmetic inequality)

$$
\begin{aligned}
&= \left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(\left(8 + \sum_{l=1}^{4}(c_l-2)\right)^2 + 3\left(\sum_{l=1}^{4}(c_l-2)\right)^2\right)\\
&= \left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(\left(\sum_{l=1}^{4} c_l\right)^2 + 3\left(\sum_{l=1}^{4}(c_l-2)\right)^2\right)\\
&\ge \left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(\sum_{l=1}^{4} c_l\right)^2 \ge 16\sqrt{c_1c_2c_3c_4}\prod_{1\le i<j\le 4} r_{ij}
\end{aligned}
$$

by A–G inequality.

Finally,

$$
\begin{aligned}
|D_4|^2 = &\ |\Re(D_4)|^2 \ge 4^4 c_1c_2c_3c_4 \prod_{1\le i<j\le 4} r_{ij}^2\\
= &\ \prod_{l=1}^{4}(4r_{ij}r_{ik}r_{jk}c_l) = \prod_{l=1}^{4}(d_3(r_{ij},r_{ik},r_{jk}) + 8r_{ij}r_{ik}r_{jk})
\end{aligned}
$$

where for each l we write $\{1,2,3,4\}\backslash\{l\} = \{i<j<k\}$. This finishes verification of Atiyah–Sutcliffe conjectures for cyclic quadrilaterals.

Proposition 2.12 *The weak Four Points Conjecture for cyclic quadrilaterals holds true.*

Proof .
From the formula obtained above we proceed along a different path

$$
\begin{aligned}
\Re(D_4) = \quad & \left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(64+16\sum_{l=1}^{4}(c_l-\cos Y^{(l)})(c_l-2)\right)\\
= \quad & \left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(4\sum_{l=1}^{4}\left[4+4(c_l-\cos Y^{(l)})(c_l-2)\right]\right)\\
= \quad & \left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(4\sum_{l=1}^{4}\left[c_l^2+(c_l-2)[3(c_l-2)+4(1-\cos Y^{(l)})]\right]\right)\\
\ge \quad & \left(\prod_{1\le i<j\le 4} r_{ij}\right)\left(4\sum_{l=1}^{4}c_l^2\right) \text{(because } 2\le c_l \text{ for each } l=1,2,3,4)\\
= \quad & \prod r_{ij}\left(\frac{1}{4}\sum_{l=1}^{4}\left(\frac{d_3(r_{ij},r_{ik},r_{jk})+8r_{ij}r_{ik}r_{jk}}{r_{ij}r_{ik}r_{jk}}\right)^2\right)
\end{aligned}
$$

and this verifies the weak Four Points Conjecture for cyclic quadrilaterals. ■

3 Almost collinear configurations. Đoković's approach

3.1 Type (A) configurations

By a type (A) configurations of N points $x_1,\dots,x_N$ we shall mean the case when $N-1$ of the points $x_1,\dots,x_N$ are collinear. Set $n=N-1$. In ([7]) Đoković has proved, for configurations of type (A), both the Atiyah conjecture (Theorem 2.1) and the first Atiyah–Sutcliffe conjecture (Theorem 3.1). By using Cartesian coordinates, with $x_i=(a_i,0)$, $a_1<a_2<\dots<a_n$ and $x_N=x_{n+1}=(0,b)'$ (with $b=1$), the normalized Atiyah matrix $M_{n+1}=M_{n+1}(\lambda_1,\dots,\lambda_n)$ (denoted by P in [7] when $b=-1$) is given by

$$
M_{n+1}=\begin{bmatrix}
1 & \lambda_1 & 0 & \cdots & 0 & 0\\
0 & 1 & \lambda_2 & \cdots & 0 & 0\\
0 & 0 & 1 & & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & & & 1 & \lambda_n\\
(-1)^n e_n & (-1)^{n-1}e_{n-1} & \cdots & \cdots & -e_1 & 1
\end{bmatrix}
$$

where $\lambda_1=a_1+\sqrt{a_1^2+b^2}<\lambda_2=a_2+\sqrt{a_2^2+b^2}<\dots<\lambda_n=a_n+\sqrt{a_n^2+b^2}$ (with $b=1$) are positive real numbers and where $e_k=e_k(\lambda_1,\dots,\lambda_n)$, $1\le k\le n$, is the k–th elementary symmetric function of $\lambda_1,\lambda_2,\dots,\lambda_n$. Its determinant

satisfies the inequality

$$\begin{aligned} D_n = &\ 1+\lambda_n e_1+\lambda_n\lambda_{n-1}e_2+\cdots+\lambda_n\lambda_{n-1}\cdots\lambda_1 e_n \\ \geq &\ 1+e_1(\lambda_1^2,\ldots,\lambda_n^2)+e_2(\lambda_1^2,\ldots,\lambda_n^2)+\cdots+e_n(\lambda_1^2,\ldots,\lambda_n^2) \\ = &\ \prod_{i=1}^n(1+\lambda_i^2) \end{aligned}$$

equivalent to the first Atiyah–Sutcliffe conjecture ([4],Conjecture 2). The second Atiyah–Sutcliffe conjecture ([4],Conjecture 3) for configurations of type (A) is equivalent to the following inequality

$$[D_{n+1}(\lambda_1,\ldots,\lambda_n)]^{n-1} \geq \prod_{k=1}^n D_n(\lambda_1,\ldots,\lambda_{k-1},\lambda_{k+1},\ldots,\lambda_n) \tag{3.26}$$

For $n=2$ this inequality takes the form

$$1+\lambda_2 e_1(\lambda_1,\lambda_2)+\lambda_1\lambda_2 e_2(\lambda_1,\lambda_2) \geq (1+\lambda_2 e_1(\lambda_2))(1+\lambda_1 e_1(\lambda_1)$$

i.e.

$$1+\lambda_2 e_1(\lambda_1,\lambda_2)+\lambda_1\lambda_2 e_2(\lambda_1,\lambda_2) \geq (1+\lambda_2^2)(1+\lambda_1^2). \tag{3.27}$$

This reduces to $(\lambda_2-\lambda_1)\lambda_1 \geq 0$, so it is true.

Even for $n=3$ the inequality (3.26) is quite messy thanks to nonsymmetric character of both sides. Knowing that sometimes it is easier to solve a more general problem we followed that path (although we didn't solve the problem in full generality). So let us start with the case $n=2$. If we look at the following inequality

$$1+X_1(\xi_1+\xi_2)+X_1X_2\xi_1\xi_2 \geq (1+X_1\xi_1)(1+X_2\xi_2)$$

which is clearly true if $X_1 \geq X_2 \geq 0$ and $\xi_1,\xi_2 \geq 0$ we obtain the inequality (3.27) simply by a specialization $X_1=\xi_1=\lambda_2$, $X_2=\xi_2=\lambda_1$. So we proceed as follows:

Let $\xi_1,\ldots,\xi_n,X_1,\ldots,X_n, n\geq 1$ be two sets of commuting indeterminates. For any $l, 1\leq l\leq n$ and any sequences $1\leq i_1\leq\cdots\leq i_l\leq n, 1\leq j_1,\ldots,j_l\leq n$ we define polynomials $\Psi_J^I=\Psi_{j_1\ldots j_l}^{i_1\ldots i_l}\in\mathbb{Q}[\xi_1,\ldots,\xi_n,X_1,\ldots,X_n]$ as follows:

$$\Psi_J^I := \sum_{k=0}^l e_k(\xi_{j_1},\xi_{j_2},\ldots,\xi_{j_l})X_{i_1}X_{i_2}\cdots X_{i_k},\ (l\geq 1),\ \Psi_\emptyset^\emptyset := 1\ (j=0)$$

where e_k is the k-th elementary symmetric function.

In particular we have

$$\begin{aligned} \Psi_j^i &= 1+\xi_j X_i, \\ \Psi_{j_1j_2}^{i_1i_2} &= 1+(\xi_{j_1}+\xi_{j_2})X_{i_1}+\xi_{j_1}\xi_{j_2}X_{i_1}X_{i_2}, \\ \Psi_{j_1j_2j_3}^{i_1i_2i_3} &= 1+(\xi_{j_1}+\xi_{j_2}+\xi_{j_3})X_{i_1}+(\xi_{j_1}\xi_{j_2}+\xi_{j_1}\xi_{j_3}+\xi_{j_2}\xi_{j_3})X_{i_1}X_{i_2}+ \\ &\quad +\xi_{j_1}\xi_{j_2}\xi_{j_3}X_{i_1}X_{i_2}X_{i_3}, \end{aligned}$$

etc.

The polynomials Ψ_J^I are symmetric w.r.t. $\xi_{j_1}, \xi_{j_2}, \ldots, \xi_{j_l}$, but nonsymmetric w.r.t. $X_{i_1}, X_{i_2}, \ldots, X_{i_l}$. By specializing X_i's to assume real values such that $X_{i_1} \geq X_{i_2} \geq \ldots \geq X_{i_l} \geq 0$ then we obtain polynomials in ξ_j's satisfying the following simple but important property.

Proposition 3.1 *(Partition property)*
Let $(I_1, \ldots, I_s)$ and $(J_1, \ldots, J_s)$ be ordered set partitions of respective sets $I = \bigcup_{p=1}^s I_p$ and $J = \bigcup_{p=1}^s J_p$ such that $|I_p| = |J_p|$, $1 \leq p \leq s$. Then the inequality

$$\Psi_J^I \geq \prod_{p=1}^s \Psi_{J_p}^{I_p}$$

holds coefficientwise w.r.t. ξ_j's.

Proof .
Proof is evident from the definition of Ψ_J^I and the monotonicity of X_i's. ■

For the powers $\left(\Psi_J^I\right)^m$ we have the following conjecture.

Conjecture 3.2 *(Weighted Multiset Partition Conjecture)*
For given natural number m and sets I and J, $|I| = |J|$, of natural numbers let $(I_1, \ldots, I_s)$ and $(J_1, \ldots, J_s)$ be the partitions of the multiset I^m consisting of m copies of all elements of I and similarly for J^m.

(i) *Then the inequality*

$$\left(\Psi_J^I\right)^m \geq \prod_{p=1}^s \Psi_{J_p}^{I_p}$$

holds coefficientwise w.r.t. ξ_j's.

(ii) *The difference*

$$\left(\Psi_J^I\right)^m - \prod_{p=1}^s \Psi_{J_p}^{I_p}$$

is multi–Schur positive with respect to partial alphabets corresponding to the atoms of the intersection lattice of the set system $\{J_1, \ldots, J_s\}$.

For example, by Partition property, we have the following inequalities

$$\Psi_{1\ldots n}^{1\ldots n} \geq \Psi_k^k \Psi_{1..\widehat{k}..n}^{1..\widehat{k}..n}, \ (1 \leq k \leq n)$$

which imply the following inequality

$$\left(\Psi_{1\ldots n}^{1\ldots n}\right)^n \geq \prod_{k=1}^n \Psi_k^k \prod_{k=1}^n \Psi_{1..\widehat{k}..n}^{1..\widehat{k}..n}$$

By Partition property we also have the following inequality

$$\Psi_{1\ldots n}^{1\ldots n} \geq \prod_{k=1}^{n} \Psi_k^k$$

The last two inequalities suggest the validity of the following inequality

$$\left(\Psi_{1\ldots n}^{1\ldots n}\right)^{n-1} \geq \prod_{k=1}^{n} \Psi_{1..\widehat{k}..n}^{1..\widehat{k}..n}$$

which is far from obvious (see Conjecture 3.3 below) although it would be a simple consequence of our Weighted Multiset Partition Conjecture.

This last conjectural inequality is interesting because it generalizes some special cases of not yet proven conjectures of Atiyah and Sutcliffe on configurations of points in three dimensional Euclidean space.

Our conjecture reads as follows:

Conjecture 3.3 *For any $n \geq 1$, let $X_1 \geq X_2 \geq \ldots \geq X_n \geq 0$, $\xi_1, \xi_2, \ldots, \xi_n \geq 0$, be nonnegative real numbers. Then we have coefficientwise (w.r.t. $\xi_1, \xi_2, \ldots, \xi_n$) inequality*

$$\left(\Psi_{12\cdots n}^{12\cdots n}\right)^{n-1} \geq \prod_{k=1}^{n} \Psi_{12\cdots\hat{k}\cdots n}^{12\cdots\hat{k}\cdots n}$$

where $12\cdots\hat{k}\cdots n$ denotes the sequence $12\cdots(k-1)(k+1)\cdots n$. The equality obviously holds true iff $X_1 = X_2 = \cdots = X_n$.

This Conjecture implies the strongest Atiyah–Sutcliffe's conjecture for almost collinear configurations of points (all but one point are collinear, called type(A) in [7]).

To illustrate the Conjecture (3.3) we consider first the cases $n = 2$ and $n = 3$.

Case $n = 2$: We have

$$\begin{aligned}\Psi_{12}^{12} &= 1 + (\xi_1 + \xi_2)X_1 + \xi_1\xi_2 X_1 X_2 = \\ &= 1 + \xi_1 X_1 + \xi_2 X_2 + \xi_1\xi_2 X_1 X_2 + (X_1 - X_2)\xi_2 = \\ &= (1 + \xi_1 X_1)(1 + \xi_2 X_2) + \xi_2(X_1 - X_2) \geq \\ &\geq (1 + \xi_1 X_1)(1 + \xi_2 X_2) = \Psi_1^1 \Psi_2^2.\end{aligned}$$

Case $n = 3$: We first write Ψ_{123}^{123} in two different ways:

$$\Psi_{123}^{123} = \xi_2(X_1 - X_2) + \widehat{\Psi}_{1\underline{2}3}^{123} \quad \text{and} \quad \Psi_{123}^{123} = \xi_3(X_1 - X_2) + \widehat{\Psi}_{12\underline{3}}^{123}.$$

Note that $\widehat{\Psi}_{1\underline{2}3}^{123}$ is obtained from Ψ_{123}^{123} by replacing the linear term $\xi_2 X_1$ by $\xi_2 X_2$, hence all its coefficients are nonnegative.

The left hand side of the Conjecture (3.3) L_3 can be rewritten as follows:

$$\begin{aligned} L_3 = (\Psi^{123}_{123})^2 &= (\xi_2(X_1 - X_2) + \widehat{\Psi}^{123}_{1\underline{2}3})\Psi^{123}_{123} \\ &= \xi_2(X_1 - X_2)\Psi^{123}_{123} + \widehat{\Psi}^{123}_{1\underline{2}3}\Psi^{123}_{123} \\ &= \xi_2(X_1 - X_2)\Psi^{123}_{123} + \widehat{\Psi}^{123}_{1\underline{2}3}(\xi_3(X_1 - X_2) + \widehat{\Psi}^{123}_{12\underline{3}}) \\ &= L'_3(X_1 - X_2) + \widehat{\Psi}^{123}_{1\underline{2}3}\widehat{\Psi}^{123}_{12\underline{3}} \end{aligned}$$

where $L'_3 = \xi_2\Psi^{123}_{123} + \xi_3\widehat{\Psi}^{123}_{1\underline{2}3}$ is a positive polynomial.
Now we have

$$L_3 \geq \widehat{L}_3 := \widehat{\Psi}^{123}_{1\underline{2}3}\widehat{\Psi}^{123}_{12\underline{3}}.$$

By using the formula

$$\widehat{\Psi}^{123}_{1\underline{2}3} = \Psi^{12}_{13} + \xi_2 X_2 \Psi^{13}_{13} = (\Psi^2_2 - 1)\Psi^{13}_{13} + \Psi^{12}_{13}$$

we can rewrite $\widehat{L}_3$ as

$$\begin{aligned} \widehat{L}_3 &= \left[(\Psi^{12}_{13} - \Psi^{13}_{13}) + \Psi^2_2\Psi^{13}_{13}\right]\widehat{\Psi}^{123}_{12\underline{3}} \\ &= \xi_1\xi_3 X_1(X_2 - X_3)\widehat{\Psi}^{123}_{12\underline{3}} + \Psi^{13}_{13}(\Psi^2_2\widehat{\Psi}^{123}_{12\underline{3}}) \end{aligned}$$

The last term in parenthesis can be written as

$$\begin{aligned} \Psi^2_2\widehat{\Psi}^{123}_{12\underline{3}} &= \Psi^{12}_{12}\Psi^{23}_{23} + \Psi^1_2(\Psi^{22}_{23} - \Psi^{23}_{23}) \\ &= \Psi^{12}_{12}\Psi^{23}_{23} + \xi_2\xi_3 X_2(X_2 - X_3)\Psi^1_2, \end{aligned}$$

so we get

$$\widehat{L}_3 = L''_3(X_2 - X_3) + \Psi^{12}_{12}\Psi^{13}_{13}\Psi^{23}_{23}$$

where L''_3 denotes the positive polynomial

$$L''_3 = \xi_1\xi_3 X_1\widehat{\Psi}^{123}_{12\underline{3}} + \xi_2\xi_3 X_2\Psi^1_2\Psi^{13}_{13}.$$

We now have an explicit formula for L_3:

$$L_3 = L'_3(X_1 - X_2) + L''_3(X_2 - X_3) + \Psi^{12}_{12}\Psi^{13}_{13}\Psi^{23}_{23}$$

with L'_3, L''_3 positive polynomials, which together with $X_1 \geq X_2 \geq X_3 (\geq 0)$ implies that

$$L_3 \geq R_3 := \Psi^{12}_{12}\Psi^{13}_{13}\Psi^{23}_{23}$$

and the Conjecture (3.3) ($n = 3$) is proved.

In fact we have proven an instance $n = 3$ $\widehat{L}_3 \geq R_3$ of a stronger conjecture which we are going to formulate now. Let $2 \leq k \leq n$. We define the modified polynomials $\widehat{\Psi}^{12\ldots k\ldots n}_{12\ldots \underline{k}\ldots n}$ as follows:

$$\widehat{\Psi}^{12\ldots k\ldots n}_{12\ldots \underline{k}\ldots n} := \xi_k(X_2 - X_1) + \Psi^{12\ldots n}_{12\ldots n}$$

obtained from $\Psi^{12\ldots n}_{12\ldots n}$ by replacing only one term $\xi_k X_1$ by $\xi_k X_2$, hence $\widehat{\Psi}^{12\ldots k\ldots n}_{12\ldots \underline{k}\ldots n}$ are still positive. Let us introduce the following notation:

$$\widehat{L}_n := \prod_{k=2}^{n} \widehat{\Psi}^{12\ldots k\ldots n}_{12\ldots \underline{k}\ldots n}\ ; \quad R_n := \prod_{k=1}^{n} \Psi^{12\ldots \hat{k}\ldots n}_{12\ldots \hat{k}\ldots n}.$$

Then clearly $L_n := (\Psi^{12\ldots n}_{12\ldots n})^{n-1} \geq \widehat{L}_n$. Now our stronger conjecture reads as

Conjecture 3.4

$$\widehat{L}_n \geq R_n \ (n \geq 1)$$

with equality iff $X_2 = X_3 = \cdots = X_n$.

More generally, we conjecture that the difference $\widehat{L}_n - R_n$ is a polynomial in the differences $X_2 - X_3,\ X_3 - X_4,\ \ldots,\ X_{n-1} - X_n$ with coefficients in $\mathbb{Z}_{\geq 0}[X_1, \ldots, X_n, \xi_1, \ldots, \xi_n]$.

Proposition 3.5

$$L_n = L'_n(X_1 - X_2) + \widehat{L}_n$$

for some positive polynomial L'_n.

Proof of Proposition 3.5.

$$\begin{aligned}
L_n &= (\Psi^{12\cdots n}_{12\cdots n})^{n-1} = (\xi_2(X_1 - X_2) + \widehat{\Psi}^{12\cdots n}_{1\underline{2}\cdots n})(\Psi^{12\cdots n}_{12\cdots n})^{n-2} \\
&= \xi_2(X_1 - X_2)(\Psi^{12\cdots n}_{12\cdots n})^{n-2} + \widehat{\Psi}^{12\cdots n}_{1\underline{2}\cdots n}(\xi_3(X_1 - X_2) + \widehat{\Psi}^{123\cdots n}_{12\underline{3}\cdots n})(\Psi^{12\cdots n}_{12\cdots n})^{n-3} \\
&= \xi_2(X_1 - X_2)(\Psi^{12\cdots n}_{12\cdots n})^{n-2} + \xi_3(X_1 - X_2)\widehat{\Psi}^{12\cdots n}_{1\underline{2}\cdots n}(\Psi^{12\cdots n}_{12\cdots n})^{n-3} + \\
&\quad + \widehat{\Psi}^{12\cdots n}_{1\underline{2}\cdots n}\widehat{\Psi}^{123\cdots n}_{12\underline{3}\cdots n}(\Psi^{12\cdots n}_{12\cdots n})^{n-3} \\
&\ \vdots \\
&= (\textstyle\sum_{k=1}^{n-1} \xi_{k+1}(\prod_{j=2}^{k} \widehat{\Psi}^{12\ldots j\ldots n}_{12\ldots \underline{j}\ldots n})(\Psi^{12\ldots n}_{12\ldots n})^{n-k})(X_1 - X_2) + \prod_{j=2}^{n} \widehat{\Psi}^{12\ldots j\ldots n}_{12\ldots \underline{j}\ldots n}.
\end{aligned}$$

■

Now we turn to study the quotient

$$\frac{L_n}{R_n} = \frac{(\Psi^{1\ldots n}_{1\ldots n})^{n-1}}{\displaystyle\prod_{k=1}^{n} \Psi^{1\ldots \widehat{k}\ldots n}_{1\ldots \widehat{k}\ldots n}}$$

by studying the growth behaviour of quotients of its factors $\Psi^{1\ldots n}_{1\ldots n}/\Psi^{1\ldots \widehat{k}\ldots n}_{1\ldots \widehat{k}\ldots n}$ w.r.t. any of its arguments X_r, $1 \leq r \leq n$.

In the following theorem we obtain an explicit formula for the numerators of the derivatives w.r.t. $X_r, (1 \le r \le n, r \neq k)$ of the quantities $\Psi^{1\ldots n}_{1\ldots n}/\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n}$. From this formulas we get some monotonicity properties which enable us to state some new (refined) conjectures later on.

Theorem 3.6 *Let*

$$\Delta_r := \partial_{X_r}\Psi^{1\ldots n}_{1\ldots n}\cdot\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n} - \Psi^{1\ldots n}_{1\ldots n}\cdot\partial_{X_r}\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n}, \quad (1 \le r \le n). \tag{3.28}$$

Then we have the following explicit formulas

(i) *for any r, $1 \le r < k(\le n)$ we have*

$$\begin{aligned}\Delta_r = \; & \xi_k \sum_{0\le i<r\le j\le n} s^{(k)}_{(2^i 1^{j-i-1})} X_1^2\cdots X_i^2 X_{i+1}\cdots\widehat{X}_k\cdots X_j + \\ & + \sum_{0\le i<r,k\le j<n} e_i e_j^{(k)} X_1^2\cdots X_i^2 X_{i+1}\cdots\widehat{X}_r\cdots\widehat{X}_k\cdots X_j(X_k - X_{j+1})\end{aligned}$$

(ii) *for any r, $(1 \le)k < r \le n$ we have*

$$\begin{aligned}\Delta_r = \; & -\Bigg(\sum_{0\le i<r\le j\le n} s^{(k)}_{(2^i 1^{j-i-1})} X_1^2\cdots X_i^2 X_{i+1}\cdots\widehat{X}_k\cdots\widehat{X}_r\cdots X_j + \\ & + \sum_{0\le i<k,r\le j<n} e_i^{(k)} e_j X_1^2\cdots X_i^2 X_{i+1}\cdots\widehat{X}_k\cdots\widehat{X}_r\cdots X_j(X_{j+1} - X_k)\Bigg)\end{aligned}$$

where $s^{(k)}_\lambda$ denotes the λ–th Schur function of $\xi_1,\ldots,\xi_{k-1},\xi_{k+1},\ldots,\xi_n$ (ξ_k omitted).

Proof of Theorem 3.6.

(i) For any r, $1 \le r < k(\le n)$ we find explicitly a formula as follows. We shall use notations $X_{1..i} := X_1X_2\cdots X_i$, for multilinear monomials and $e_i := e_i(\xi_1,\ldots,\xi_n)$, $e_i^{(k)} = e_i(\xi_1,\ldots,\widehat{\xi_k},\ldots\xi_n)$ for the elementary symmetric functions (here k is fixed). Then we can rewrite our basic quantities

$$\Psi^{1\ldots n}_{1\ldots n} := \sum_{i=0}^{n} e_i X_{1..i} \tag{3.29}$$

$$\begin{aligned}\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n} \; := \; & \sum_{i=0}^{k-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k}\sum_{i=k}^{n-1} e_i^{(k)} X_{1..i+1} = \\ = \; & \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k}\sum_{i=k}^{n-1} e_i^{(k)} X_{1..i}(X_{i+1} - X_k)\end{aligned} \tag{3.30}$$

For the derivatives we get immediately

$$\partial_{X_r}\Psi^{1\ldots n}_{1\ldots n} = \frac{1}{X_r}\sum_{i=r}^{n} e_i X_{1..i} = \frac{1}{X_r}\left(\Psi^{1\ldots n}_{1\ldots n} - \sum_{i=0}^{r-1} e_i X_{1..i}\right) \tag{3.31}$$

$$\partial_{X_r}\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n} = \frac{1}{X_r}\sum_{i=r}^{n-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k X_r}\sum_{i=k}^{n-1} e_i^{(k)} X_{1..i}(X_{i+1} - X_k) \quad (3.32)$$

$$= \frac{1}{X_r}\left(\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n} - \sum_{i=0}^{r-1} e_i^{(k)} X_{1..i}\right) \quad (3.33)$$

By plugging (3.31) and (3.33) into (3.28) we obtain

$$X_r\Delta_r = \Psi^{1\ldots n}_{1\ldots n}\left(\sum_{i=0}^{r-1} e_i^{(k)} X_{1..i}\right) - \Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n}\left(\sum_{i=0}^{r-1} e_i X_{1..i}\right) =$$

and after simple cancelation, by invoking (3.30) we get

$$= \left(\sum_{j=r}^{n} e_j X_{1..j}\right)\left(\sum_{i=0}^{r-1} e_i^{(k)} X_{1..i}\right) - \\ \left(\sum_{j=r}^{n-1} e_j^{(k)} X_{1..j} + \frac{1}{X_k}\sum_{j=k}^{n-1} e_j^{(k)} X_{1..j}(X_{j+1} - X_k)\right)\left(\sum_{i=0}^{r-1} e_i X_{1..i}\right)$$

i.e.

$$X_r\Delta_r = \sum_{0\le i<r\le j\le n}(e_j e_i^{(k)} - e_i e_j^{(k)})X_{1..i}X_{1..j} + \frac{1}{X_k}\sum_{0\le i<r, k\le j<n} e_i e_j^{(k)} X_{1..i}X_{1..j}(X_k - X_{j+1})$$

If we use a simple identity $e_j = e_j^{(k)} + \xi_k e_{j-1}^{(k)}$, we can identify the quantity

$$e_j e_i^{(k)} - e_i e_j^{(k)} = (e_j^{(k)} + \xi_k e_{j-1}^{(k)})e_i^{(k)} - (e_i^{(k)} + \xi_k e_{i-1}^{(k)})e_j^{(k)} =$$

$$= \begin{vmatrix} e_{j-1}^{(k)} & e_j^{(k)} \\ e_{i-1}^{(k)} & e_i^{(k)} \end{vmatrix}\xi_k = s^{(k)}_{2^i 1^{j-i-1}}\xi_k$$

Thus in this case $(1 \le r < k)$ we obtain a formula

$$\Delta_r = \xi_k \sum_{0\le i<r\le j\le n} s^{(k)}_{(j-1,i)} X_1^2\cdots X_i^2 X_{i+1}\cdots\widehat{X}_k\cdots X_j + \\ + \sum_{0\le i<r,k\le j<n} e_i e_j^{(k)} X_1^2\cdots X_i^2 X_{i+1}\cdots\widehat{X}_r\cdots\widehat{X}_k\cdots X_j(X_k - X_{j+1})$$

(where $e_j^{(k)} = e_j^{(k)} = e_j(\xi_1,\ldots,\widehat{\xi_k},\ldots,\xi_n)$) in terms of Schur functions (of arguments $\xi_1,\ldots,\widehat{\xi_k},\ldots,\xi_n$) corresponding to a transpose $(2^i 1^{j-i-1})$ of a partition $(j-1,i)$ (cf. Jacobi–Trudi formula, I 3.5 in [9]).

(ii) For any r, $(1 \le) k < r \le n$. In this case we use

$$\partial_{X_r}\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n} = \frac{1}{X_k X_r}\sum_{j=r-1}^{n-1} e_j^{(k)} X_{1..j+1}$$

$$\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n} = \sum_{i=0}^{k-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k}\sum_{i=k}^{n-1} e_i^{(k)} X_{1..i+1} =$$
$$= \frac{1}{X_k}\left(\sum_{i=0}^{k-1} X_{1..i}(X_k - X_{i+1}) + \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i}\right)$$

By plugging this into (3.28) we get

$$\begin{aligned}
X_k X_r \Delta_r \quad &= \left(\sum_{j=r}^{n} e_j X_{1..j}\right)\left(\sum_{i=0}^{k-1} e_i^{(k)} X_{1..i}(X_k - X_{i+1}) + \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i+1}\right) - \\
&\quad - \left(\sum_{j=0}^{r-1} e_j X_{1..j} + \sum_{j=r}^{n} e_j X_{1..j}\right)\left(\sum_{i=r-1}^{n-1} e_i^{(k)} X_{1..i+1}\right) \\
&= \left(\sum_{i=0}^{r-2} e_i^{(k)} X_{1..i+1}\right)\left(\sum_{j=r}^{n} e_j X_{1..j}\right) - \left(\sum_{i=0}^{r-1} e_i X_{1..i}\right)\left(\sum_{j=r-1}^{n-1} e_j^{(k)} X_{1..j+1}\right) + \\
&\quad + \sum_{i=0}^{k-1}\sum_{j=r}^{n} e_i^{(k)} e_j X_{1..i} X_{1..j}(X_k - X_{i+1}) \\
&= \left(\sum_{i=1}^{r-1} e_{i-1}^{(k)} X_{1..i}\right)\left(\sum_{j=r}^{n} e_j X_{1..j}\right) - \left(\sum_{i=0}^{r-1} e_i X_{1..i}\right)\left(\sum_{j=r}^{n} e_{j-1}^{(k)} X_{1..j}\right) + \\
&\quad + \sum_{i=0}^{k-1}\sum_{j=r}^{n} e_i^{(k)} e_j X_{1..i} X_{1..j}(X_k - X_{i+1})
\end{aligned}$$

By using a formula for elementary symmetric functions ($e_i = e_i^{(k)} + \xi_k e_{i-1}^{(k)}$) we can write in terms of Schur functions (of arguments $\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n$), where λ' is a conjugate of λ.

$$e_{i-1}^{(k)} e_j - e_i e_{j-1}^{(k)} = e_{i-1}^{(k)} e_j^{(k)} - e_i^{(k)} e_{j-1}^{(k)} = -\begin{vmatrix} e_{j-1}^{(k)} & e_j^{(k)} \\ e_{i-1}^{(k)} & e_i^{(k)} \end{vmatrix} = -s^{(k)}_{2^i 1^{j-i-1}} = -s^{(k)}_{(j-1,i)'}$$

Thus we obtain a formula

$$\begin{aligned}
\Delta_r = \quad &- \Bigg(\sum_{0\le i<r\le j\le n} s^{(k)}_{(j-1,i)'} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots \widehat{X}_r \cdots X_j + \\
&\quad + \sum_{0\le i<k, r\le j<n} e_i^{(k)} e_j X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots \widehat{X}_r \cdots X_j (X_{j+1} - X_k)\Bigg)
\end{aligned}$$

Corollary 3.7 *(X_r–monotonicity)*
Let $X_1 \geq \cdots \geq X_n \geq 0$, $\xi_1, \ldots, \xi_n \geq 0$ be as before. Then

(i) for any r, $1 \leq r < k$ $(\leq n)$ we have

$$\frac{\Psi^{1\dots n}_{1\dots n}}{\Psi^{1\dots\widehat{k}\dots n}_{1\dots\widehat{k}\dots n}} \geq \frac{\Psi^{1\dots\ r+1\ r+1\ \dots n}_{1\dots\ r\ \ r+1\ \dots n}}{\Psi^{1\dots\ r+1\ r+1\ \dots\widehat{k}\dots n}_{1\dots\ r\ \ r+1\ \dots\widehat{k}\dots n}}$$

(ii) for any r, $(1 \leq)$ $k < r$ $(\leq n)$ we have

$$\frac{\Psi^{1\dots n}_{1\dots n}}{\Psi^{1\dots\widehat{k}\dots n}_{1\dots\widehat{k}\dots n}} \geq \frac{\Psi^{1\dots\ r-1\ r-1\ \dots n}_{1\dots\ r-1\ \ r\ \dots n}}{\Psi^{1\dots\widehat{k}\dots\ r-1\ r-1\ \dots n}_{1\dots\widehat{k}\dots\ r-1\ \ r\ \dots n}}$$

Now we illustrate how to use Corollary 3.7 to prove our Conjecture 3.3 for $n = 2, 3, 4$ and 5.

Case $n = 2$

$$Q_2 := \frac{\Psi^{12}_{12}}{\Psi^1_1\Psi^2_2} \geq \frac{\Psi^{22}_{12}}{\Psi^2_1\Psi^2_2} = 1 \text{ (by } (i))$$

Case $n = 3$

$$\begin{aligned} Q_3 := \frac{\Psi^{123}_{123}\Psi^{123}_{123}}{\Psi^{12}_{12}\Psi^{13}_{13}\Psi^{23}_{23}} &\geq \frac{\Psi^{223}_{123}\Psi^{123}_{123}}{\Psi^{22}_{12}\Psi^{13}_{13}\Psi^{23}_{23}} \geq \frac{\Psi^{223}_{123}\Psi^{223}_{123}}{\Psi^{22}_{12}\Psi^{13}_{13}\Psi^{23}_{23}} \text{ (by } (i)) \\ &\geq \frac{\Psi^{222}_{123}\Psi^{223}_{123}}{\Psi^{22}_{12}\Psi^{22}_{13}\Psi^{23}_{23}} \geq \frac{\Psi^{222}_{123}\Psi^{222}_{123}}{\Psi^{22}_{12}\Psi^{22}_{13}\Psi^{23}_{23}} = 1 \text{ (by } (ii)) \end{aligned}$$

Case $n = 4$

$$Q_4 := \frac{(\Psi^{1234}_{1234})^3}{\Psi^{123}_{123}\Psi^{124}_{124}\Psi^{134}_{134}\Psi^{234}_{234}} \geq \cdots \geq \frac{\Psi^{2244}_{1234}(\Psi^{2224}_{1234})^2}{\Psi^{224}_{123}\Psi^{224}_{124}\Psi^{224}_{134}\Psi^{224}_{234}} \quad (\geq 1)$$

This last inequality follows from the following symmetric function identity:

$$\begin{aligned} &\Psi^{2244}_{1234}(\Psi^{2224}_{1234})^2 - \Psi^{224}_{123}\Psi^{224}_{124}\Psi^{224}_{134}\Psi^{224}_{234} = \\ &X_2^2X_4^4m_{2222} + 2X_2^2X_4^3m_{2221} + X_2^2X_4^2m_{222} + 3X_2^2X_4^2m_{2211} + X_2^2X_4m_{221} \\ &+4X_2^2X_4m_{2111} + X_2^2m_{211} + X_2(3X_2 + 2X_4)m_{1111} + X_2m_{111} \end{aligned}$$

where $m_\lambda = m_\lambda(\xi_1, \xi_2, \xi_3, \xi_4)$ are the monomial symmetric functions.

Case $n = 5$

$$Q_5 := \frac{(\Psi^{1\dots5}_{1\dots5})^4}{\prod_{k=1}^5 \Psi^{1\dots\widehat{k}\dots5}_{1\dots\widehat{k}\dots5}} \geq \cdots \geq \frac{(\Psi^{22244}_{12345}\Psi^{22444}_{12345})^2}{\Psi^{2244}_{1234}\Psi^{2244}_{1235}\Psi^{2244}_{1245}\Psi^{2244}_{1345}\Psi^{2244}_{2345}} \quad (\geq 1)$$

The last inequality is equivalent to an explicit symmetric function identity with all coefficients (w.r.t. monomial basis) positive.

Now we state our stronger conjecture.

Conjecture 3.8 *(for symmetric functions)*
Let $X_1 \geq X_2 \geq \cdots \geq X_n \geq 0$ and $\xi_1, \dots, \xi_n \geq 0$. Then the inequalities

(a) For n even

$$\Psi^{2\ 2\ 4\ 4 \ldots n\ n}_{1\ 2\ \ldots\ n-1\ n} \left(\prod_{k=1}^{n/2} \Psi^{2\ 2\ 4\ 4 \ldots 2k\ 2k\ 2k \ldots n-2\ n-2\ n}_{1\ 2\ 3\ 4\ \ldots\ n-1\ n} \right)^2 \geq \prod_{k=1}^{n} \Psi^{2\ 2\ 4\ 4 \ldots n-2\ n-2\ n}_{1\ 2\ \ldots\ \widehat{k}\ \ldots\ n-1\ n}$$

(b) For n odd

$$\left(\prod_{k=1}^{\lfloor n/2 \rfloor} \Psi^{2\ 2\ 4\ 4 \ldots 2k\ 2k\ 2k \ldots n-1\ n-1}_{1\ 2\ 3\ 4\ \ldots\ n-1\ n} \right)^2 \geq \prod_{k=1}^{n} \Psi^{2\ 2\ 4\ 4 \ldots n-1\ n-1}_{1\ 2\ \ldots\ \widehat{k}\ \ldots\ n}$$

hold true coefficientwise (m–positivity).

Now we motivate another inequalities for symmetric functions which also refine the strongest Atiyah–Sutcliffe conjecture for configurations of type (A). Let $n = 3$. We apply Corollary 3.7 by using steps (ii) only.

$$Q_3 := \frac{\Psi^{123}_{123}\Psi^{123}_{123}}{\Psi^{12}_{12}\Psi^{13}_{13}\Psi^{23}_{23}} \geq \frac{\Psi^{113}_{123}\Psi^{123}_{123}}{\Psi^{12}_{12}\Psi^{13}_{13}\Psi^{13}_{23}} \geq \frac{\Psi^{112}_{123}\Psi^{123}_{123}}{\Psi^{12}_{12}\Psi^{12}_{13}\Psi^{13}_{23}} \geq \frac{\Psi^{112}_{123}\Psi^{122}_{123}}{\Psi^{12}_{12}\Psi^{12}_{13}\Psi^{12}_{23}} \geq 1$$

The last inequality is equivalent to nonnegativity of the expression

$$\Psi^{112}_{123}\Psi^{122}_{123} - \Psi^{12}_{12}\Psi^{12}_{13}\Psi^{12}_{23} \ \ (= X_1(X_1 - X_2)^2\xi_1\xi_2\xi_3 \geq 0).$$

Similarly, for $n = 4$, the symmetric function inequality stronger than $Q_4 \geq 1$ would be the following

$$\Psi^{1123}_{1234}\Psi^{1223}_{1234}\Psi^{1233}_{1234} \geq \Psi^{123}_{123}\Psi^{123}_{124}\Psi^{123}_{134}\Psi^{123}_{234}$$

Now we state a general conjecture for symmetric functions which imply the strongest Atiyah–Sutcliffe conjecture for almost collinear type (A) configurations.

Conjecture 3.9 *Let $X_1 \geq \cdots \geq X_n \geq$, $\xi_1, \ldots \xi_n \geq 0$. Then the following inequality for symmetric functions in $\xi_1, \ldots, \xi_n$*

$$\Psi^{112 \ldots n-1}_{123 \ldots n}\Psi^{1223 \ldots n-1}_{1234 \ldots n} \cdots \Psi^{12 \ldots n-2\ n-1\ n-1}_{12 \ldots n-2\ n-1\ n} \geq \Psi^{1\ 2 \ldots n-1}_{1\ 2 \ldots n-1}\Psi^{1\ 2 \ldots n-1}_{1\ 2 \ldots n-2\ n} \cdots \Psi^{1\ 2 \ldots n-1}_{2\ 3 \ldots n-1}$$

i.e.

$$\prod_{k=1}^{n-1} \Psi^{1\ 2 \ldots k\ \ k\ \ldots n}_{1\ 2 \ldots k\ k+1 \ldots n} \geq \prod_{k=1}^{n} \Psi^{1\ 2\ \ldots\ n-1}_{1\ 2 \ldots \widehat{k} \ldots n}$$

holds true coefficientwise (m–positivity).

Remark 3.10 *Conjectures 3.8 and 3.9 seems to hold also for the Schur basis of symmetric functions in $\xi_1, \ldots, \xi_n$.*

We have checked this Conjecture 3.9 up to $n = 5$ by using `Maple` and symmetric function package `SF` of J. Stembridge. For n bigger than five the computations are extremely intensive and hopefully in the near future would be possible by using more powerful computers.

Note that the right hand side of the Conjecture 3.9 involves symmetric functions of partial alphabets $\xi_1, \xi_2, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n$. But the left hand side doesn't have this "defect". Our objective now is to give explicit formula for the right hand side in terms of the elementary symmetric functions of the full alphabet $\xi_1, \xi_2, \ldots, \xi_n$. This we are going to achieve by using resultants as follows.

Lemma 3.11 *For any k, $(1 \leq k \leq n)$, we have*

$$\Psi^{1\ldots k\ldots n-1}_{1\ldots\widehat{k}\ldots n} = \sum_{j=0}^{n-1} a_j \xi_k^{n-1-j}$$

where

$$\begin{aligned}
&a_{n-1} = 1 + X_1 e_1 + X_1 X_2 e_2 + \ldots + X_1 \cdots X_{n-1} e_{n-1}, \\
&a_{n-2} = -X_1 - X_1 X_2 e_1 - \ldots - X_1 \cdots X_{n-1} e_{n-2}, \\
&\cdots \\
&a_0 = (-1)^{n-1} X_1 \cdots X_{n-1}
\end{aligned}$$

i.e.

$$a_{n-1-j} = (-1)^j \sum_{i=j}^{n-1} X_1 \cdots X_i e_{i-j}$$

Proof of Lemma 3.11.
By definition we have

$$\Psi^{1\ldots n-1}_{1\ldots\widehat{k}\ldots n} = \sum_{i=0}^{n-1} X_1 \cdots X_i e_i^{(k)} \tag{3.34}$$

where $e_i^{(k)}$ is the i–th elementary function of $\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n$. Now from the decomposition

$$(1 + \xi_k t)^{-1} \prod_{j=1}^{n} (1 + \xi_j t) = \prod_{j \neq k} (1 + \xi_j t) = \sum_{i=0}^{n-1} e_i^{(k)} t^i$$

we get

$$e_i^{(k)} = e_i - e_{i-1}\xi_k + e_{i-2}\xi_k^2 - \cdots + (-1)^i \xi_k^i$$

By substituting this into equation (3.34) the Lemma 3.11 follows. ■

Then, by Lemma 3.11, the right hand side

$$R_n = \prod_{k=1}^{n} \Psi^{1\ 2\ \dots\ k\ \dots\ n-1}_{1\ 2\ \dots\ \widehat{k}\ \dots\ \ n} = \prod_{k=1}^{n} \left(\sum_{j=0}^{n-1} a_j \xi_k^{n-1-j} \right)$$

can be understood as a resultant $R_n = Res(f,g)$ of the following two polynomials

$$\begin{aligned} f(x) &= \sum_{j=0}^{n-1} a_j x^{n-1-j} \\ g(x) &= \prod_{i=1}^{n} (x-\xi_i) = \sum_{j=0}^{n} (-1)^j e_j x^{n-j} \end{aligned}$$

The Sylvester formula

$$R_n = \begin{vmatrix} 1 & -e_1 & e_2 & -e_3 & \dots & (-1)^n e_n & & & \\ & 1 & -e_1 & e_2 & -e_3 & \dots & & & \\ & & \ddots & & & & & & \\ & & & 1 & -e_1 & \cdots & & & \\ a_0 & a_1 & a_2 & \cdots & a_n & & & & \\ & a_0 & a_1 & a_2 & \cdots & a_n & & & \\ & & \ddots & & & & & & \\ & & & a_0 & a_1 & a_2 & & \cdots & a_n \end{vmatrix} \quad \left(=: \begin{vmatrix} A & B \\ C & D \end{vmatrix} \right)$$

can be simplified as

$$= |A| \cdot |D - CA^{-1}B| = |D - CA^{-1}B|.$$

The entries of the $n \times n$ matrix $\Delta := D - CA^{-1}B$ are given by

$$\delta_{ij} = \begin{cases} (-1)^{j-i-1} \displaystyle\sum_{k=j+1}^{n} X_1 \cdots X_{k+i-j} e_k, & 0 \le i < j \le n-1 \\ (-1)^{j-i} \displaystyle\sum_{k=0}^{j} X_1 \cdots X_{k+i-j} e_k, & 0 \le j \le i \le n-1 \end{cases}$$

For example, for $n = 3$

$$\Delta_3 = \begin{vmatrix} 1 & X_1 e_2 + X_1 X_2 e_3 & -X_1 e_3 \\ -X_1 & 1 + X_1 e_1 & X_1 X_2 e_3 \\ X_1 X_2 & -X_1 - X_1 X_2 e_1 & 1 + X_1 e_1 + X_1 X_2 e_2 \end{vmatrix}$$

By elementary operations we get

$$\Delta_3 = \begin{vmatrix} 1 & * & * \\ 0 & \Psi^{112}_{123} & X_1(X_2 - X_1) e_3 \\ 0 & X_2 - X_1 & \Psi^{122}_{123} \end{vmatrix} = \begin{vmatrix} \Psi^{112}_{123} & X_1(X_2 - X_1) e_3 \\ X_2 - X_1 & \Psi^{122}_{123} \end{vmatrix}$$

Similarly, for $n = 4$ we obtain

$$\Delta_4 = \begin{vmatrix} \Psi_{1234}^{1123} & -X_1(X_1-X_2)e_3 - X_1X_2(X_1-X_3)e_4 & X_1(X_1-X_2)e_4 \\ -(X_1-X_2) & \Psi_{1234}^{1223} & -X_1X_2(X_2-X_3)e_4 \\ X_1(X_2-X_3) & -(X_1-X_3) - X_1(X_2-X_3)e_1 & \Psi_{1234}^{1233} \end{vmatrix}$$

In general

$$\Delta_n = \det(\delta'_{ij})_{1 \le i,j \le n-1}$$

where

$$\delta'_{ij} = \begin{cases} (-1)^{j-i} \displaystyle\sum_{k=j+1}^{n} X_1 \cdots X_{k+i-j-1}(X_i - X_{k+i-j})e_k & , 1 \le i < j \le n-1 \\ \Psi_{1\ 2\ \dots\ n}^{1\ \dots\ i\ i\ \dots\ n} & , i = j \\ (-1)^{j-i} \displaystyle\sum_{k=0}^{j} X_1 \cdots X_{k+i-j-1}(X_{k+i-j} - X_i)e_k & , 1 \le j < i \le n-1 \end{cases}$$

Corollary 3.12 *The conjecture 3.9 is equivalent to a Hadamard type inequality, holding coefficientwise, for the (non Hermitian) matrix* $(\delta'_{ij})_{1 \le i,j \le n-1}$*, i.e.*

$$\prod_{i=1}^{n-1} \delta'_{ii} \ge \det(\delta'_{ij})$$

4 Verification of the Đoković's strengthening of the Atiyah–Sutcliffe Conjecture (C2) for some nonplanar configurations with dihedral symmetry

Here we basically follow Đoković's [8], where only Atiyah conjecture C1 was proved, make some additional refinements including a proof of Atiyah–Sutcliffe conjecture C2.

Let $N = m + n$ points be such that

1. The first m points $x_1, \dots, x_m$ lie on a line L.
2. The remaining n points $y_j = x_{m+j+1}$ $(j = 0, 1, ..., n-1)$ are the vertices oif a regular n–gon whose plane is perpendicular to L and whose centroid lies on L.

We may assume $L = \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{C} = \mathbb{R}^3$ and write $x_i = (a_i, 0)$, $1 \le i \le m$, $a_1 \le \ldots \le a_m$ and $y_j = (0, b_j)$, $b_j = -\xi^j$, $\xi = e^{2\pi i/n}$, $0 \le j \le n-1$.

We set

$$\lambda_i = a_i + \sqrt{1 + a_i^2}$$

Recall that $a_1 < \cdots < a_m$ and, consequently $0 < \lambda_1 < \cdots < \lambda_m$. Then the associated polynomials p_i (up to scalar factors) are given by

$$p_i(x, y) = x^{m-i} y^{i-1}(x^n - \lambda_i^n y^n),\ 1 \leq i \leq m$$
$$p_{m+j+1}(x, y) = \prod_{s \neq j} \left(x + \frac{\overline{b_s} - \overline{b_j}}{|b_s - b_j|} y \right) \cdot \prod_{i=1}^{m} (y - \lambda_i b_j x), 0 \leq j < n$$

By noting that

$$b_s - b_j = 2i\xi^{\frac{j+s}{2}} \sin \frac{\pi(j - s)}{n}$$

(in Đoković ξ^{j+s} should be replaced by $\xi^{\frac{j+s}{2}}$) we obtain

$$x + \frac{\overline{b_s} - \overline{b_j}}{|b_s - b_j|} y = \left(-\overline{b_j} y - i\xi^{\frac{s-j}{2}} \operatorname{sgn}(s - j) \right) \frac{1 - \overline{b_s} b_j}{|b_s - b_j|}$$

and

$$y - \lambda_i b_j x = -b_j(-\overline{b_j} y + \lambda_i x)$$

Note also that

$$\{\xi^{\frac{s-j}{2}} \operatorname{sgn}(s - j) | s = 1, \ldots, j - 1, j + 1, \ldots, n\} = \{e^{\pi i k/n} | k = 1, \ldots, n - 1\}$$

Thus, after dehomogenizing the polynomials p_i by setting $x = 1$, we obtain (up to scalar factors) the following polynomials:

$$\widetilde{P}_i(y) = y^{i-1}(1 - \lambda_i^n y^n),\ 1 \leq i \leq m;$$
$$\widetilde{P}_{m+j+1}(y) = f(\xi^{-1} y),\ 0 \leq j < n$$

where

$$f(y) = \prod_{s=1}^{n-1} (y - ie^{\pi i s/n}) \prod_{i=1}^{m} (y + \lambda_i)$$

(in Đoković the last n polynomials are reordered)

The main result of Đoković is the Theorem *3.1* where he proved Atiyah conjecture for configurations described above, by explicitly computing the determinant of the coefficients matrix $\widetilde{P}$ of the polynomials $\{\widetilde{p}_k(y) | k = 1, \ldots, \underbrace{m + n}_{N}\}$ in terms of the coefficients of

$$f(y) = \sum_{k=0}^{N-1} \widetilde{E}_k y^{N-1-k}$$

His formula reads as follows:

$$\left|\det(\widetilde{P})\right| = n^{n/2}\prod_{k=0}^{n-1} f_k$$

where

$$f_k = \sum_{s\geq 0}\left(\prod_{j=1}^{s}\lambda^n_{N-jn-k}\right)\widetilde{E}_{k+sn},\ 0\leq k<n.$$

We shall now present an amazingly simple formula for coefficients of the polynomial

$$h(y) := \prod_{s=1}^{n-1}(y-ie^{\pi is/n}) = \sum_{j=0}^{n-1} c_j y^{n-1-j}$$

Proposition 4.1 *let* $\gamma_k := \cot\left(\frac{k\pi}{2n}\right)$. *Then*

$$c_0 = 1,\ c_j = \prod_{k=1}^{j}\gamma_k \qquad (1\leq j\leq n-1)$$

Proof .
Put $\xi_k = -ie^{\pi ik/n}$, $k=1,\ldots,n-1$. Then

$$\begin{aligned} c_j &= \text{the j–th elementary symmetric function of } \xi_1,\ldots,\xi_{n-1}\\ &= e_j(\xi_1,\ldots,\xi_{n-1}) \end{aligned}$$

Let us first compute the power sums

$$\begin{aligned} p_s = \sum_{k=1}^{n-1}\xi_k^s = (-i)^s\sum_{k=1}^{n-1}e^{\pi isk/n} &= (-i)^s(e^{\pi is/n}-e^{\pi is})/(1-e^{\pi is})\\ &= \begin{cases} (-1)^{\frac{s}{2}-1}, s \text{ even}\\ (-1)^{\frac{s-1}{2}}\cot(\frac{s\pi}{2n}) = (-1)^{\frac{s-1}{2}}\gamma_s, s \text{ odd}\end{cases} \end{aligned}$$

The proof will be by induction. For $j=1$ we have $c_1 = \xi_1+\cdots+\xi_{n-1} = p_1 = \gamma_1$. Suppose that the proposition is true for all $k<i$. Then by Newton formula for symmetric functions

$$je_j = \sum_{k=1}^{j}(-1)^{k-1}p_k e_{j-k} = \sum_{l=1}^{\lceil j/2\rceil}(p_{2l-1}e_{j-2l+1}-p_{2l}e_{j-2l})$$

we obtain by writing $c_{j-2l+1} = c_{j-2l}\gamma_{j-2l+1}$

$$\begin{aligned}
je_j &= \sum_{l=1}^{\lceil j/2\rceil} \left((-1)^{l-1}\gamma_{2l-1}\gamma_{j-2l+1} - (-1)^{l-1}\right) c_{j-2l} \\
&= \sum_{l=1}^{\lceil j/2\rceil} (-1)^{l-1}(\gamma_{2l-1}\gamma_{j-2l+1} - 1)c_{j-2l} \\
&\overset{*}{=} \sum_{l=1}^{\lceil j/2\rceil} (-1)^{l-1}(\gamma_{2l-1} + \gamma_{j-2l+1})\gamma_j c_{j-2l} \\
&= \sum_{l=1}^{\lceil j/2\rceil} (p_{2l-1}c_{j-2l} - p_{2l-2}\gamma_{j-2l+1}c_{j-2l})\gamma_j \quad \text{(here } p_0 := -1\text{)} \\
&= \sum_{l=1}^{\lceil j/2\rceil} (p_{2l-1}c_{j-2l} - p_{2l-2}c_{j-2l+1})\gamma_j \\
&= \sum_{l=1}^{\lceil j/2\rceil} (p_{2l-1}c_{j-1-(2l-1)} - p_{2l-2}c_{j-1-(2l-2)})\gamma_j \\
&= (-p_0c_{j-1} + \sum_{l=1}^{\lceil (j-1)/2\rceil} (p_{2l-1}c_{j-1-(2l-1)} - p_{2l}c_{j-1-2l)})\gamma_j \\
&\overset{**}{=} (c_{j-1} + (j-1)c_{j-1})\gamma_j \\
&= jc_{j-1}\gamma_j = jc_j
\end{aligned}$$

Here in $(*)$ we have used the cotangent addition formula $\cot(\alpha)\cot(\beta) - 1 = (\cot\alpha + \cot\beta)\cot(\alpha+\beta)$ and in $(**)$ Newton formula for $i-1$ which holds by induction hypothesis. The proposition is thus proved. ■

For our dihedral configurations we can state the stronger conjecture of Atiyah and Sutcliffe ([8], Conjecture *2.*) as follows

$$n^{\frac{n}{2}} \prod_{k=0}^{n-1} f_k \geq 2^{\binom{n}{2}} \prod_{i=0}^{n} (1+\lambda_i^2)^n \tag{4.35}$$

where

$$f_k = \sum_{s\geq 0} \left(\prod_{j=1}^{s} \lambda_{N-jn-k}^n \widetilde{E}_{k+sn}, \quad (0 \leq k < n) \right) \tag{4.36}$$

From the factorization

$$f(y) = h(y) \prod_{i=1}^{m} (y+\lambda_i)$$

we can write

$$\widetilde{E}_k = \sum_{i=0}^{n-1} c_i E_{k-i}$$

in terms of elementary symmetric functions $E_k = e_k(\lambda_1, \dots, \lambda_m)$ of our positive quantities $0 < \lambda_1 < \cdots < \lambda_m$ with coefficients c_i given in Proposition 4.1 (note that $c_0 = 1 \le c_1 \le \cdots \le c_{\lfloor \frac{n-1}{2} \rfloor} \ge \cdots \ge c_{n-1} = 1$ (unimodality) and $c_i = c_{n-1-i}$ (symmetry)).

Now we shall prove a generalization of the Đoković's conjecture which apparently strengthens (4.35).

Theorem 4.2 *We have:*

1. $\displaystyle\prod_{k=0}^{n-1} f_k \ge \prod_{k=0}^{n-1} c_k \left(\sum_{l=0}^{m} \left(\prod_{j=0}^{l-1} \lambda_{m-j} E_l \right) \right)^n$

2. $\displaystyle\prod_{k=0}^{n-1} f_k \ge \prod_{k=0}^{n-1} c_k \prod_{i=1}^{m} (1 + \lambda_i^2)^n$

Proof .
Let us write

$$f_k = \sum_{l=0}^{m} \varphi_{kl} E_l$$

Let us substitute $\widetilde{E}_{k+sn} = \displaystyle\sum_{i=0}^{n-1} c_i E_{k-i+sn}$ into (4.36). Then for fixed k ($0 \le k < n-1$) and given l ($0 \le l \le m$) we seek $s \ge 0$ and i, $0 \le i < n$ such that $l = k - 1 + sn$, i.e. $l - k = sn - i$, $0 \le i < n$. We conclude that s and i are uniquely determined by a division algorithm (with nonpositive remainder):

$$s_k := \left\lceil \frac{l-k}{n} \right\rceil, \quad i_k = s_k n - l - k.$$

Hence

$$\varphi_{kl} = \prod_{j=1}^{s_k} \lambda_{N-jn-k}^n c_{i_k}$$

with s_k and i_k just defined. It is easy to see that

$$s_k = s_0 \left(= \left\lceil \frac{l}{n} \right\rceil \right) \text{ and } i_k = i_0 + k \text{ for } 0 \le k \le n - i_0 - 1$$

and

$$s_k = s_0 - 1 \text{ and } i_k = i_0 + k - n \text{ for } n - i_0 \le k \le n-1.$$

Lemma 4.3 *For each l, $0 \leq l \leq m$, we have*

$$\prod_{k=0}^{n-1} \varphi_{kl} = \prod_{j=0}^{l-1} \lambda_{m-j}^{n} \prod_{j=0}^{n-1} c_j$$

Proof (of Lemma).

$$\begin{aligned}\prod_{k=0}^{n-1} \varphi_{kl} &= \prod_{k=0}^{n-i_0-1} \left(\prod_{j=1}^{s_0} \lambda_{N-jn-k}^{n} \prod_{k=i_0}^{n-1} c_k \right) \prod_{k=n-i_0}^{n-1} \prod_{j=1}^{s_0-1} \lambda_{N-jn-k}^{n} \prod_{k=0}^{i_0-1} c_k \\ &= \prod_{k=0}^{n-1} \prod_{j=1}^{s_0-1} \lambda_{N-jn-k}^{n} \prod_{k=0}^{n-i_0-1} \lambda_{N-s_0 n-k}^{n} \prod_{k=0}^{n-1} c_k\end{aligned}$$

We put now $N = n + m$

$$\begin{aligned}&= \lambda_m^n \lambda_{m-1}^n \cdots \lambda_{m+n-s_0 n-(n-i_0-1)}^n \prod_{k=0}^{n-1} c_k \\ &= \lambda_m^n \lambda_{m-1}^n \cdots \lambda_{m-l+1}^n \prod_{k=0}^{n-1} c_k\end{aligned}$$

■

Proof (of Theorem).
We shall use the Hölder inequality

$$\begin{aligned}\prod_{k=0}^{n-1} f_k = \prod_{k=0}^{n-1} \left(\sum_{l=0}^{m} \varphi_{kl} E_l \right) &\geq \left(\sum_{l=0}^{m} \left(\prod_{k=1}^{n-1} \varphi_{kl} E_l \right)^{\frac{1}{n}} \right)^n \\ &= \left(\sum_{l=0}^{m} \prod_{j=0}^{l-1} \lambda_{m-j} \left(\prod_{j=0}^{n-1} c_j \right)^{\frac{1}{n}} E_l \right)^n \quad \text{(by lemma)} \\ &= \left(\prod_{j=0}^{n-1} c_j \right) \left(\sum_{l=0}^{m} \prod_{j=0}^{l-1} \lambda_{m-j} E_l \right)^n\end{aligned}$$

Thus *1.* is proved. To obtain *2.* we apply Đoković proof of Atiyah conjecture for type A configurations

$$\sum_{l=0}^{m} \prod_{j=0}^{l-1} \lambda_{m-j} E_l \geq \prod_{i=1}^{m} \left(1 + \lambda_i^2\right)$$

(c.f. section 3.) ■

5 Appendix

After the first version of this paper was finished, in the meantime, we have discovered formulas for the partial derivatives, of the quantities $\Psi_{1\ldots n}^{1\ldots n}/\Psi_{1\ldots\widehat{k}\ldots n}^{1\ldots\widehat{k}\ldots n}$,

with respect to variables ξ_r (Note that in Theorem 3.6 we have given formulas w.r.t. variables X_r!).

Lemma 5.1 *For $2 \le r \le n$ the partial derivative w.r.t. ξ_r of the quotient $\Psi^{1\dots n}_{1\dots n}/\Psi^{2\dots n}_{2\dots n}$ is given by*

$$\left(\Psi^{2\dots n}_{2\dots n}\right)^2 \partial_{\xi_r}\left(\frac{\Psi^{1\dots n}_{1\dots n}}{\Psi^{2\dots n}_{2\dots n}}\right) = \sum_{i\ge j} s'_{ij} X_1 (X_2\cdots X_j)^2 X_{j+1}\cdots X_{i+1}(X_{j+1}-X_{i+2})$$

where s'_{ij} is the conjugated Schur function $s_{ij} = s_{ij}(\xi_2,\dots,\xi_{r-1},\xi_{r+1},\dots,\xi_n)$ corresponding to a two–rowed partition $\lambda = (i \ge j)$.
In particular for $X_1 \ge \cdots \ge X_n > 0$ the function $\Psi^{1\dots n}_{1\dots n}/\Psi^{2\dots n}_{2\dots n}$ is monotonically increasing w.r.t. the variable ξ_r (for $r = 1$, too).

Proof .
By using the formula $\Psi^{1\dots n}_{1\dots n} = \Psi^{1\dots n-1}_{1\dots\widehat{r}\dots n} + X_1\xi_r\Psi^{2\dots n}_{1\dots\widehat{r}\dots n}$ we get

$$\partial_{\xi_r}(\Psi^{1\dots n}_{1\dots n})\Psi^{2\dots n}_{2\dots n} - \Psi^{1\dots n}_{1\dots n}\partial_{\xi_r}(\Psi^{2\dots n}_{2\dots n}) =$$
$$= X_1\Psi^{2\dots n}_{1\dots\widehat{r}\dots n}\left(\Psi^{2\dots n-1}_{2\dots\widehat{r}\dots n} + X_2\xi_r\Psi^{3\dots n}_{2\dots\widehat{r}\dots n}\right) - \left(\Psi^{1\dots n-1}_{1\dots\widehat{r}\dots n} + X_1\xi_r\Psi^{2\dots n}_{1\dots\widehat{r}\dots n}\right)X_2\Psi^{3\dots n}_{2\dots\widehat{r}\dots n}$$
$$= X_1\Psi^{2\dots n}_{1\dots\widehat{r}\dots n}\Psi^{2\dots n-1}_{2\dots\widehat{r}\dots n} - X_2\Psi^{1\dots n-1}_{1\dots\widehat{r}\dots n}\Psi^{3\dots n}_{2\dots\widehat{r}\dots n}$$
$$= X_1\left(\Psi^{2\dots n-1}_{2\dots\widehat{r}\dots n} + X_2\xi_1\Psi^{3\dots n}_{2\dots\widehat{r}\dots n}\right)\Psi^{2\dots n-1}_{2\dots\widehat{r}\dots n} - X_2\left(\Psi^{1\dots n-2}_{2\dots\widehat{r}\dots n} + X_1\xi_1\Psi^{2\dots n-1}_{2\dots\widehat{r}\dots n}\right)\Psi^{3\dots n}_{2\dots\widehat{r}\dots n}$$
$$= X_1\left(\Psi^{2\dots n-2}_{2\dots\widehat{r}\dots n}\right)^2 - X_2\Psi^{1\dots n-2}_{2\dots\widehat{r}\dots n}\Psi^{3\dots n}_{2\dots\widehat{r}\dots n}$$

With $e_i = e_i^{(1r)} = e_i(\xi_2,\dots,\xi_{r-1},\xi_{r+1},\dots,\xi_n)$ denoting the i–th elementary symmetric function of the truncated alphabet $A^{(1r)} = \{\xi_2,\dots,\xi_{r-1},\xi_{r+1},\dots,\xi_n\}$ we have further

$$= X_1\left(\sum_{i,j} e_ie_jX_{2\dots i+1}X_{2\dots j+1}\right) - X_2\left(\sum_{i,j} e_ie_jX_{1\dots i}X_{3\dots j+2}\right)$$
$$= \sum_{i,j} e_ie_jX_{1..i+1}X_{2..j+1} - \sum_{i,j} e_ie_jX_{1..i}X_{2..j+2}$$
$$= \sum_{i,j}\begin{vmatrix} e_i & e_{i+1}\\ e_{j-1} & e_j\end{vmatrix} X_{1..i+1}X_{2..j+1}$$
$$= \sum_{i\ge j}\begin{vmatrix} e_i & e_{i+1}\\ e_{j-1} & e_j\end{vmatrix} X_1(X_{2..j})^2X_{j+1}\cdots X_{i+1}(X_{j+1}-X_{i+2})$$

Now by Jacobi–Trudy formula we can write $\begin{vmatrix} e_i & e_{i+1}\\ e_{j-1} & e_j\end{vmatrix}$ as the conjugated Schur function $s'_{ij} = {s'}^{(1r)}_{ij}$ corresponding to a partition $(i \ge j)$. ■

Corollary 5.2 *(ξ_n–monotonicity)*
We have the following inequality:

$$\frac{\Psi^{1\dots n}_{1\dots n}}{\Psi^{2\dots n}_{2\dots n}} \ge \frac{\Psi^{1\dots n-1}_{1\dots n-1}}{\Psi^{2\dots n-1}_{2\dots n-1}}$$

Proof .
By Lemma 5.1 by letting $\xi_n \downarrow 0$ we get

$$\Psi^{1\ldots n}_{1\ldots n}/\Psi^{2\ldots n}_{2\ldots n} \geq \left.\Psi^{1\ldots n}_{1\ldots n}/\Psi^{2\ldots n}_{2\ldots n}\right|_{\xi_n=0} = \Psi^{1\ldots n-1}_{1\ldots n-1}/\Psi^{2\ldots n-1}_{2\ldots n-1}$$

■

By using this Corollary we state a strengthening of our Conjecture 3.3:

Conjecture 5.3

$$\left(\Psi^{1\ldots n}_{1\ldots n}\right)^{n-2} \geq \Psi^{2\ldots n-1}_{2\ldots n-2} \prod_{k=2}^{n-1} \Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n}$$

We also have formulas for partial derivative of the quotient $\Psi^{1\ldots n}_{1\ldots n}/\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n}$ w.r.t. variable ξ_r, $2 \leq r \leq n$, which are more complicated than for $k = 1$ (given in Lemma 5.1). Without loss of generality we take $r = n$ and proceed as follows:

$$\begin{aligned}
&\partial_{\xi_n}(\Psi^{1\ldots n}_{1\ldots n})\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n} - \Psi^{1\ldots n}_{1\ldots n}\partial_{\xi_n}(\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n}) = \\
&= X_1\Psi^{2\ldots n}_{1\ldots n-1}\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n} - X_1\Psi^{1\ldots n}_{1\ldots n}\Psi^{2\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n-1} \\
&= X_1\Psi^{2\ldots n}_{1\ldots n-1}\left(\Psi^{1\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n} + X_1\xi_n\Psi^{2\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n-1}\right) - X_1\left(\Psi^{1\ldots n-1}_{1\ldots n-1} + X_1\xi_n\Psi^{2\ldots n}_{1\ldots n-1}\right)\Psi^{2\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n-1} \\
&= X_1\left(\Psi^{2\ldots n}_{1\ldots n-1}\Psi^{1\ldots\widehat{k}\ldots n-1}_{1\ldots\widehat{k}\ldots n-1} - \Psi^{1\ldots n-1}_{1\ldots n-1}\Psi^{2\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n-1}\right) \\
&= X_1\left[\left(\Psi^{2\ldots n-1}_{1\ldots\widehat{k}\ldots n-1} + X_2\xi_k\Psi^{3\ldots n}_{1\ldots\widehat{k}\ldots n-1}\right)\Psi^{1\ldots\widehat{k}\ldots n-1}_{1\ldots\widehat{k}\ldots n-1} - \right. \\
&\qquad \left. - \left(\Psi^{1\ldots n-2}_{1\ldots\widehat{k}\ldots n-1} + X_1\xi_k\Psi^{2\ldots n-1}_{1\ldots\widehat{k}\ldots n-1}\right)\Psi^{2\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n-1}\right] \\
&= X_1\left[\Psi^{2\ldots n-1}_{1\ldots\widehat{k}\ldots n-1}\Psi^{1\ldots\widehat{k}\ldots n-1}_{1\ldots\widehat{k}\ldots n-1} - \Psi^{1\ldots n-2}_{1\ldots\widehat{k}\ldots n-1}\Psi^{2\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n-1} + \right. \\
&\qquad \left. + \xi_k\left(X_2\Psi^{3\ldots n}_{1\ldots\widehat{k}\ldots n-1}\Psi^{1\ldots\widehat{k}\ldots n-1}_{1\ldots\widehat{k}\ldots n-1} - X_1\Psi^{2\ldots n-1}_{1\ldots\widehat{k}\ldots n-1}\Psi^{2\ldots\widehat{k}\ldots n}_{1\ldots\widehat{k}\ldots n-1}\right)\right] \\
&= X_1\left[I_1 - \xi_k I_2\right]
\end{aligned}$$

Now we first compute

$$
\begin{aligned}
&I_1 = \Psi^{2\dots n-1}_{1\dots\widehat{k}\dots n-1}\Psi^{1\dots\widehat{k}\dots n-1}_{1\dots\widehat{k}\dots n-1} - \Psi^{1\dots n-2}_{1\dots\widehat{k}\dots n-1}\Psi^{2\dots\widehat{k}\dots n}_{1\dots\widehat{k}\dots n-1} = \\
&\left(\sum_{i=0}^{k-2} e_i X_{2..i+1} + \sum_{i=k-1}^{n-2} e_i X_{2..i+1}\right)\left(\sum_{j=0}^{k-1} e_j X_{1..j} + \sum_{j=k}^{n-2} e_j X_{1..\widehat{k}..j+1}\right) - \\
&-\left(\sum_{j=0}^{k-1} e_j X_{1..j} + \sum_{j=k}^{n-2} e_j X_{1..j}\right)\left(\sum_{i=0}^{k-2} e_i X_{2..i+1} + \sum_{i=k-1}^{n-2} e_i X_{2..\widehat{k}..i+2}\right) = \\
&= \sum_{i=k-1}^{n-2}\sum_{j=0}^{k-1} e_i e_j \left(X_{2..i+1}X_{1..j} - X_{2..\widehat{k}..i+1}X_{1..j}\right) + \\
&+ \sum_{j=k}^{n-2}\sum_{i=0}^{k-2} e_j e_i \left(X_{1..\widehat{k}..j+1}X_{2..i+1} - X_{1..j}X_{2..i+1}\right) + \\
&+ \sum_{i=k-1}^{n-2}\sum_{j=k}^{n-2} e_i e_j \left(X_{2..i+1}X_{1..\widehat{k}..j+1} - X_{1..j}X_{2..\widehat{k}..i+2}\right)
\end{aligned}
$$

By replacing, in the middle sum, j with $i+1$ and i with $j-1$, and observing that then $X_{1..\widehat{k}..i+2}X_{2..j} - X_{1..i+1}X_{2..j} = -(X_{2..i+1}X_{1..j} - X_{2..\widehat{k}..i+2}X_{1..j})$ the contribution of the first two sums is

$$
\sum_{i=k-1}^{n-2}\sum_{j=0}^{k-1}\begin{vmatrix} e_i & e_{i+1} \\ e_{j-1} & e_j \end{vmatrix} X_{2..\widehat{k}..i+1}(X_k - X_{i+2})X_{1..j}
$$

The third sum can similarly be transformed to the following form:

$$
\sum_{k\le j\le i\le n-2}\begin{vmatrix} e_i & e_{i+1} \\ e_{j-1} & e_j \end{vmatrix} X_{2..\widehat{k}..i+1}(X_{j+1} - X_{i+2})X_{1..j}
$$

Hence

$$
I_1 = \sum_{0\le j,\max\{j,k-1\}\le i\le n-2} s'_{ij} X_{2..\widehat{k}..i+1}(X_{\max\{j+1,k\}} - X_{i+2})X_{1..j} \quad (\ge 0)
$$

By a similar manipulation we can obtain the expression for the quantity

$$
\begin{aligned}
&I_2 = X_1\Psi^{2\dots n-1}_{1\dots\widehat{k}\dots n-1}\Psi^{2\dots\widehat{k}\dots n}_{1\dots\widehat{k}\dots n-1} - X_2\Psi^{3\dots n}_{1\dots\widehat{k}\dots n-1}\Psi^{1\dots\widehat{k}\dots n-1}_{1\dots\widehat{k}\dots n-1} = \\
&= X_1 - X_2 + \sum_{i=1}^{n-1}\sum_{j\le min\{k-1,i\}} s'_{ij} X_{2..\widehat{k}..i+2}X_{1..j}(X_{j+1} - X_k) \ge 0
\end{aligned}
$$

where s'_{ij} is conjugated Schur function $s'_{ij} = {s'}^{(kn)}_{ij}$. We see that

$$
\left(\Psi^{1\dots\widehat{k}\dots n}_{1\dots\widehat{k}\dots n}\right)^2 \partial_{\xi_n}\left(\frac{\Psi^{1\dots n}_{1\dots n}}{\Psi^{1\dots\widehat{k}\dots n}_{1\dots\widehat{k}\dots n}}\right) = X_1\left[I_1 - \xi_k I_2\right]
$$

has both positive and negative terms. And we have not been able to apply it so far.

Now we illustrate use of ξ–monotonicity (in addition to X–monotonicity) for proving once more the case $n=4$ of our Conjecture 3.3:

$$
\begin{aligned}
&\frac{(\Psi^{1234}_{1234})^3}{\Psi^{234}_{234}\Psi^{134}_{134}\Psi^{124}_{124}\Psi^{123}_{123}} = \frac{\Psi^{1234}_{1234}}{\Psi^{234}_{234}\Psi^{123}_{123}}\frac{\Psi^{1234}_{1234}}{\Psi^{134}_{134}}\frac{\Psi^{1234}_{1234}}{\Psi^{124}_{124}} \geq \ (by\ \xi_4\text{–monotonicity}) \\
&\geq \frac{1}{\Psi^{23}_{23}}\frac{\Psi^{1234}_{1234}}{\Psi^{134}_{134}}\frac{\Psi^{1234}_{1234}}{\Psi^{124}_{124}} \geq \ (by\ X_1\text{–monotonicity twice and } X_4\text{–monotonicity}) \\
&\geq \frac{1}{\Psi^{23}_{23}}\frac{\Psi^{2234}_{1243}}{\Psi^{234}_{143}}\frac{\Psi^{2233}_{1234}}{\Psi^{223}_{124}} \geq \ (by\ \xi_3\text{–monotonicity}) \\
&\geq \frac{1}{\Psi^{23}_{23}}\frac{\Psi^{223}_{124}}{\Psi^{23}_{14}}\frac{\Psi^{2233}_{1234}}{\Psi^{223}_{124}} = \frac{\Psi^{2233}_{1234}}{\Psi^{23}_{23}\Psi^{23}_{14}} \geq 1
\end{aligned}
$$

Similarly the cases $n=5,6,7$ of Conjecture 3.3 would be, by using ξ–monotonicity and X–monotonicity, consequences of the following inequalities

$$\widetilde{Q}_n \geq 1$$

where

$$\widetilde{Q}_5 = \Psi^{22344}_{12345}\Psi^{22344}_{12345}/\Psi^{234}_{234}\Psi^{234}_{135}\Psi^{2244}_{1245}$$

$$\widetilde{Q}_6 = \Psi^{223445}_{123456}\Psi^{233455}_{123456}/\Psi^{2345}_{2345}\Psi^{2345}_{1346}\Psi^{2345}_{1256}$$

$$\widetilde{Q}_7 = \Psi^{2234556}_{1234567}\Psi^{2334566}_{1234567}\Psi^{2344566}_{1234567}/\Psi^{23456}_{23456}\Psi^{23456}_{13457}\Psi^{23456}_{12467}\Psi^{234566}_{123567}$$

5.1 Computer verification of the Conjecture 3.3 (and hence of the Atiyah–Sutcliffe conjecture C3) for almost collinear $9+1$ configuration.

Let us now explain our computer verification of the inequality $\widetilde{Q}_9 \geq 1$ where

$$\widetilde{Q}_9 = \frac{\Psi^{223456778}_{123456789}\Psi^{233456788}_{123456789}\Psi^{223456678}_{123456789}\Psi^{234456788}_{123456789}}{\Psi^{2345678}_{2345678}\Psi^{2345678}_{1345679}\Psi^{2345678}_{1245689}\Psi^{2345678}_{1235789}\Psi^{22346788}_{12346789}}$$

which refines the case $n=9$ of the Conjecture 3.3. We have observed first that $\widetilde{Q}_9$ is symmetric in partial alphabets

$$A_1 = \{\xi_1, \xi_2, \xi_8, \xi_9\}, \ A_2 = \{\xi_3, \xi_4, \xi_6, \xi_7\}, \ A_3 = \{\xi_5\}$$

then by introducing the elementary symmetric functions $\{e_1, e_2, e_3, e_4\}$ of A_1 and $\{f_1, f_2, f_3, f_4\}$ of A_2 we first computed the products

$$\Psi^{2345678}_{2345678}\Psi^{2345678}_{1345679} \text{ and } \Psi^{2345678}_{1245689}\Psi^{2345678}_{1235789}$$

in terms of $\{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4, \xi_5\}$. Then by successive application of Stembridge's `Maple SF` package we expressed the difference $\Delta := numer(\widetilde{Q}_9) -$

$denom(\widetilde{Q}_9)$ in terms of the Schur functions of both alphabets A_1 and A_2. Then we factored each coefficient in such a multi–Schur expansion and into non-monomial factors we substituted $X_2 = X_3 + h_2$, $X_3 = X_4 + h_3$, …, $X_7 = X_8 + h_7$. Then the computation showed that the coefficients of all monomials in $X_8, h_2, \ldots, h_7$ were nonnegative. The factoring out the trivial monomial factors in $X_2, \ldots, X_8$ (which are trivially nonnegative) was crucial because otherwise the expansion of multi–Schur function coefficients in terms of increments $h_2, \ldots, h_7$ may not be feasible.

6 Appendix 2

Here we first recall a remarkable inequality of I. Schur (c.f. J. Michael Steele: The Cauchy–Schwarz Master Class, Cambridge University Press, 2004.)

For all values $x, y, z \geq 0$ and all $\alpha \geq 0$ we have

$$\begin{aligned} I_\alpha(x,y,z) &:= \sum x^\alpha(x-y)(x-z) = \\ &= x^\alpha(x-y)(x-z) + y^\alpha(y-x)(y-z) + z^\alpha(z-x)(z-y) \geq 0 \end{aligned}$$

with equality iff either $x = y = z$ or two of the variables are equal and the third is zero. Note that I_α is a symmetric function. For a proof we can assume $0 \leq x \leq y \leq z$. Then clearly $x^\alpha(x-y)(x-z) \geq 0$ and by grouping the other two terms we get $(z-y)[z^\alpha(z-x) - y^\alpha(y-x)] \geq 0$ by observing that $z \geq y$ and $z - x \geq y - x$.

Now we state and prove several properties of a function

$$d_3(x,y,z) := (x+y-z)(x-y+z)(-x+y+z) \quad (x,y,z \geq 0)$$

which frequently appears in the main part of this paper.

We note that the area $A = A(a,b,c)$ of a triangle with sides lengths a, b, c is given, according to the Heron-s formula:

$$\begin{aligned} (4A)^2 &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\ &= (a+b+c)d_3(a,b,c) \\ &= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 \end{aligned}$$

Properties of the function d_3:

Proposition 6.1 *We have the following identities and inequalities:*

1. $xyz - d_3(x,y,z) = \sum x(x-y)(x-z) \geq 0$
2. $d_3(x,y,z)^2 - d_3(x^2,y^2,z^2) =$
   $= \sum x^2(y^2 - yz + z^2 - x^2)^2 + \left(\sum x(y^2 - yz + z^2 - x^2)\right)^2 \geq 0$
3. $d_3(x,y,z)^2 - d_3(x^2,y^2,z^2) =$
   $= 8x^2y^2z^2 - 2(xyz + x^3 + y^3 + z^3)d_3(x,y,z) \geq 0$

4. $(x+y+z)^2 d_3(x,y,z)^2 - 3(x^2+y^2+z^2) d_3(x^2,y^2,z^2) =$
$= 4\sum x^4(x^2-y^2)(x^2-z^2) \geq 0$

5. $(x+y+z)(X+Y+Z)d_3(x,y,z)d_3(X,Y,Z) - 3(xX+yY+zZ)d_3(xX,yY,zZ) =$
$2\sum(x^2(x^2-y^2)X^2(X^2-Z^2) + X^2(X^2-Y^2)x^2(x^2-z^2)) + (x^2(Y^2-Z^2) +$
$y^2(Z^2-X^2) + z^2(X^2-Y^2))^2 \geq 0$

Proof .
All identities *1.–5.* can be easily checked by expansion. The inequality in *1.* follows from Schur's inequality ($\alpha = 1$), in *2.* it is evident since the rhs is the sum of four squares (see [5]). Case *3.* follows from *2.* Case *4.* follows from Schur's inequality ($\alpha = 2$). Case *5.* follows from a generalization of the case $\alpha = 2$ of Schur's inequality:

$$II_2(x,y,z,X,Y,Z) = \sum x(x-y)X(X-Z) =$$
$$= x(x-y)X(X-Z) + y(y-x)Y(Y-Z) + z(z-x)Z(Z-Y) \geq 0$$

(by letting $y = x+h$, $z = y+k$, $Y = X+H$, $Z = Y+K$, $h,k,H,K \geq 0$). ■

Corollary 6.2 *From the Proposition we get the following inequalities:*

$$d_3(x,y,z) \leq xyz \quad (from\ 1.\ )$$

and a stronger inequality $d_3(x,y,z) \leq 4x^2y^2z^2/(xyz + x^3 + y^3 + z^3)$ *(from 3.)*
From 2. we have the inequality

$$d_3(x,y,z)^2 \geq d_3(x^2,y^2,z^2)$$

which can also be obtained from 4. (which implies famous Finsler–Hadwiger inequality) by using the inequality $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$.
The inequality 5. , with the help of Chebyshev inequality

$$(x+y+z)(X+Y+Z) \leq 3(xX+yY+zZ) \quad (x \leq y \leq z, \quad X \leq Y \leq Z)$$

gives us the following inequality (which seems to be new):

$$d_3(x,y,z)d_3(X,Y,Z) \geq d_3(xX,yY,zZ)$$

(when $0 \leq x \leq y \leq z$, $0 \leq X \leq Y \leq Z$*).*

Remark 6.3 *If* a,b,c *are side lengths of a triangle then inequality* $d_3(a,b,c) \leq abc$ *follows also directly from the following identity*

$$abc - d_3(a,b,c) = \frac{1}{2}[(-a+b+c)(b-c)^2 + (a-b+c)(a-c)^2 + (a+b-c)(a-b)^2]$$

from which we also have the following inequality

$$8(abc - d_3(a,b,c))^3 \geq d_3(a,b,c)(a-b)^2(a-c)^2(b-c)^2$$

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